

Fibonacci numbers: A brief history and counting problems

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Preamble

It's very likely that you, dear Reader, have heard of the Fibonacci numbers, and it's likely that you like them.² But it's also likely that you don't know much about these numbers: most people don't, and there is a lot to know about them. It is therefore my pleasure here to present to you some less-known facts about these Fibonacci numbers.

What are Fibonacci numbers?

As you probably know, the Fibonacci numbers are the numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

that you get by starting with 0 and 1 and adding consecutive pairs of numbers to get the next; for instance, $0 + 1 = 1$; $1 + 1 = 2$; $1 + 2 = 3$; and so on. We use the mathematical notation $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, \dots as a useful way in which to keep track of the Fibonacci numbers. We can then calculate all Fibonacci numbers by the following recursive identity and initial conditions

$$F_n = F_{n-1} + F_{n-2} \quad \text{and} \quad F_0 = 0, F_1 = 1. \quad (1)$$

Solving this identity gives us an elegant and interestingly irrational expression for F_n , called *Binet's Formula* [10]:

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi)^{-n})$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the famous *golden ratio*, or *divine proportion*, that appears surprisingly often in different parts of maths and nature.

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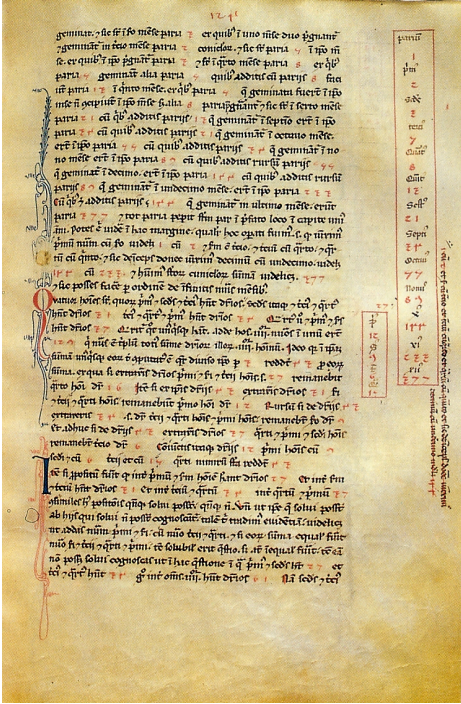
²To test this claim, I asked people on Facebook, predominantly non-mathematicians, what their favourite part of maths was, if any. The 105 replies were intriguingly varied - 65 distinct categories of answers, including "Fibonacci". Only 4 replies indicated "Fibonacci", which is not much, or 6 if the categories "golden ratio" and "Fibonacci" were merged, which is still not much. That doesn't support my claim very well but 4 replies is still among the 3rd most frequent categories. The most frequent category, with 6 replies, was "Algebra", in case you were wondering. Sorted, the frequencies look like this:



So, what is a Fibonacci?

Fibonacci, or Leonardo Fibonacci, is one of the commonly used names for Leonardo, a mathematician from Pisa who lived from 1170 to sometime between 1240 and 1250. He wrote several influential books on mathematics, including the *Liber Abaci* ("Book of Calculation") [5] from 1202. Fibonacci numbers appear in that book, shown here in the right margin of a slightly later edition of the book that now lies in the National Central Library of Florence [4].

To illustrate Fibonacci numbers, Leonardo gave several "real-world" examples of the contrived sort that mathematicians sometimes inflict on students to make maths relatable but which mostly confuses and annoys. One example was of rabbits left in an enclosure to procreate with abandon for a year, with new rabbits appearing each month and taking a month to reach sexual maturity. In this simple and slightly disturbing model, the monthly numbers of rabbits are given by Fibonacci numbers when we ignore constraints such as attraction, genetics, longevity and physics.



Rabbits and Fibonacci numbers

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New numbers

Leonardo was an excellent mathematician, one of the best and most famous in Europe at the time, and proved significant new maths, particularly in number theory. However, his main contribution to Western maths was to introduce it to Eastern maths: advanced Arabic maths, including algebra, the sophisticated geometry and proofs of Euclid, and the number 0 and the Hindu-Arabic decimal system that we use today. You can recognise decimal numbers in red on the page of the *Liber Abaci* above.

0 1 2 3 4 5 6 7 8 9

Decimals are better for calculations than the Roman numerals used in Europe at the time, but they did not catch on when first introduced there in the 10th century. 250 years passed before Leonardo re-introduced decimals to Europe, along with *Liber Abaci* as their user manual: basic arithmetic, fractions, currency and measurement conversions, profit and interest calculations, numerical approximations, and more. This made decimals accessible and many merchants, administrators and scholars adopted them. However, many others remained opposed to these un-Christian new numbers until 1500 or so, when Roman numerals ceased to be used for calculations.

Older origins

Leonardo was not the first to introduce and study Fibonacci numbers. They were also introduced and studied roughly five hundred years earlier, perhaps even as early as 200BC, by Indian scholars of Sanskrit poetry [7].

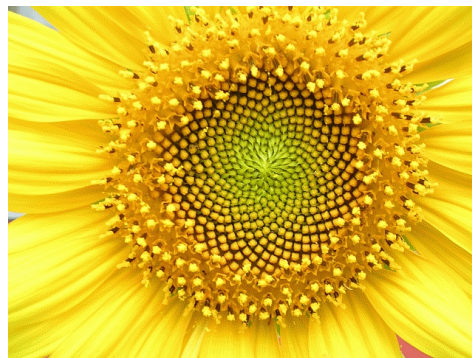
This poetry features syllables (morae) that are either one or two beats long. The scholars asked questions such as: how many syllable patterns have a given number of beats? How many have a given number of syllables? These questions are strikingly similar to those we consider in combinatorial mathematics today.

Around 700, Virahanka was able to derive the recursive identity (1) for Fibonacci numbers, and he and later Sanskrit scholars, in the 12th century, introduced other complex combinatorial concepts, including multinomial coefficients.

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Ancient origins

As Fibonacci numbers are so easy to generate, I would wager that they have been discovered countless times. Certainly, these numbers were discovered many eons ago, by Mother Nature for use in Her biotechnological designs: Fibonacci numbers and their golden ratio φ appear in flower petals, tree branches, seeds, pine cones, shells, beehive (not rabbit) population numbers and certain other naturally occurring phenomena, even in the spiralling arms of some galaxies. The reason for this is that Fibonacci numbers turn out to be the numbers required for efficient packing of certain clustering growth processes.



Fibonacci numbers in nature



Photo by L. Shyamal, used under Creative Commons cc-by-sa-2.5.

For excellent explanations of this phenomenon, I recommend Vihart's fascinating and excellent YouTube trilogy [9].

Fibonacci numbers count

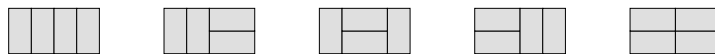
As we have now seen, Fibonacci numbers appear throughout history and nature. They also appear throughout mathematics, often as solutions to mathematical counting problems. Below are some of these problems, taken from [10] and expanded upon. In each case, the number F_n counts some number of objects. Can you see why and how? To do this, try to see how each counting problem has the same recursive identity and initial conditions as given in (1).

Tilings

The Fibonacci number F_{n+1} is the number of $1 \times n$ tilings with tiles  and , of length 1 and 2.³ For instance, there are $F_5 = 5$ tilings of size 1×4 :





Similarly, F_{n+1} is the number of $2 \times n$ tilings with tiles of length 2: . For instance, there are $F_5 = 5$ tilings of size 2×4 :



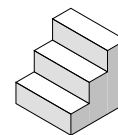
Can you find other examples of tilings that equal F_{n+1} when counted?

Sanskrit poetry and stairs

The tiling examples above show the connection between Fibonacci numbers and the Sanskrit poems studied by the ancient Indian scholars: each tiling of size $1 \times n$ with tiles  and  corresponds to a pattern of n beats with 1- and 2-beat syllables.

There are F_{n+1} such syllable patterns, which is what Virahanka proved.

There are many other variations of this counting problem. For instance, F_{n+1} is the number of ways to climb n stairs, taking 1 or 2 stairs at a time. Can you find other variations?



Binary words

How many subsets of $\{1, \dots, n\}$ contain no consecutive integers? The answer is F_{n+2} . For instance, there are $F_5 = 5$ subsets of $\{1, 2, 3\}$ with no consecutive integers:

\emptyset $\{1\}$ $\{2\}$ $\{3\}$ $\{1, 3\}$

In terms of binary words – sequences of 0s and 1s – this problem is equivalent to asking: How many binary words of length n have no consecutive 0s? For instance, there are $F_5 = 5$ binary words of length $n = 3$ with no consecutive 0s:

111 011 101 110 010

A different sort of problem is: How many binary words of length n have no odd number of consecutive 0s? The answer is F_{n+1} . For instance, there are $F_5 = 5$ binary words of length $n = 4$ with no odd number of consecutive 0s:

1111 0011 1001 1100 0000

Also, F_n is the number of binary words of length n starting with 0 and having no even number of consecutive 0s or 1s. There are $F_5 = 5$ such binary words of length 5:

01110 01010 00010 01000 00000

³To see this, let a_n count such tilings. By removing their last tile, we can uniquely match the tilings of length n to the tilings of length $n - 1$ and $n - 2$, so $a_n = a_{n-1} + a_{n-2}$. Also, $a_1 = 1 = F_2$ and $a_2 = 2 = F_3$, so the numbers a_n satisfy the same defining conditions (1) as F_{n+1} . This means that $a_n = F_{n+1}$.

More tilings

Returning to tilings, we could ask: how many $1 \times n$ tilings can be laid with tiles of any length *except* 1? The answer is F_{n-1} . For instance, there are $F_5 = 5$ such 1×6 tilings:

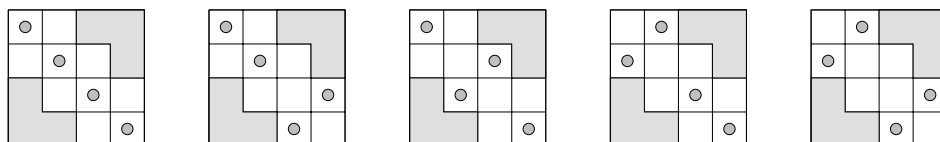


Similarly, there are F_n tilings of size $1 \times n$ that contain only tiles that have odd length. For instance, there are $F_5 = 5$ such tilings of size 1×5 :



Permutations and matrices

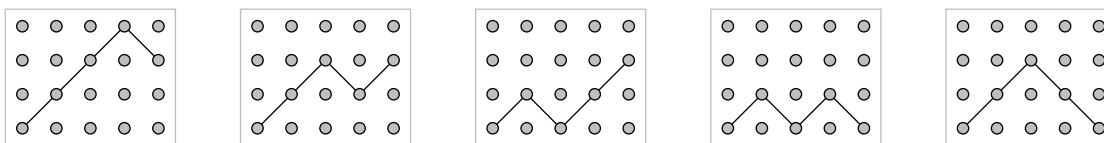
Fibonacci numbers also appear when we look at permutations: F_{n+1} is the number of ways in which we can place n dots in an $n \times n$ grid, either on the main diagonal or just next to it, so that no two dots share the same row or column. For instance, there are $F_5 = 5$ such ways to draw $n = 4$ dots in a 4×4 grid:



This counting problem sometimes appears, in disguise, when we count the terms of expanded determinants and permanents of matrices. By the way, this counting problem is essentially the same as one(s) that we have seen previously. Can you see which?

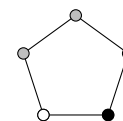
Drawing lines

Let's play a game. On a $4 \times n$ grid, start at the bottom-left hand corner and draw a line to the right-hand edge of the grid, moving either one up or one down at each step. In how many ways can you draw such lines? The answer is the Fibonacci number F_n . For instance, there are $F_5 = 5$ such ways to draw these lines when $n = 5$:



We can imagine this problem differently as a game on four dots connected sequentially: $\circ - \circ - \circ - \circ$. From the first dot, draw along the lines in $n - 1$ steps, from dot to dot.

We can play this game on other shapes, including this pentagon here. From the white dot, draw along lines in n steps. If c_n is the number of these drawn lines that end back at the white dot, and if p_n is the number of drawn lines that end at the black dot, then $F_{n+1} = p_n - c_n$. For instance, try to check that $c_4 = 6$ and $p_4 = 1$. We see that $F_5 = 5 = p_4 - c_4$.



What else?

I hope you had fun figuring out how the counting problems led to Fibonacci numbers. There are many other problems like these, and there is much more to learn about these numbers. There is even a scientific journal, *The Fibonacci Quarterly*, dedicated to new research related to them. There is also the golden ratio φ , from Binet's Formula, that the ratios F_{n+1}/F_n converge to. Just like Fibonacci numbers, φ appears countless places in nature, maths, and art. That is a topic big enough to write books about, and several such books have indeed been written, including [2, 3].

It makes me happy to see maths with such old roots flourish so strongly still today.

References

- [1] F.C. Gies, Fibonacci, Italian Mathematician, Encyclopædia Britannica, <https://www.britannica.com/biography/Fibonacci>, last updated April 12, 2020.
- [2] P. Hemenway, *Divine Proportion: Phi In Art, Nature, and Science*, New York: Sterling, 2005.
- [3] M. Livio, *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*, New York City: Broadway Books, 2003.
- [4] H. Lüneburg, *Leonardi Pisani Liber Abaci oder Lesevergnügen eines Mathematikers*, 2nd ed., Bibliographisches Institut Wissenschaftsverlag, Mannheim, 1993.
- [5] L. Pisano, *Fibonacci's Liber Abaci*, a translation into modern English of Leonardo Pisano's *Book of Calculation*, translated from the Latin and with an introduction, notes and bibliography by L.E. Sigler, Springer-Verlag, New York, 2002.
- [6] D.E. Smith and L.C. Karpinski, *The Hindu-Arabic Numerals*, The Athenæum Press, Ginn and Company, Proprietors, Boston, 1911.
- [7] P. Singh, The so-called Fibonacci numbers in ancient and medieval India, *Historia Mathematica* **12** (1985), 229–244.
- [8] H.K. Sørensen, *Tal*, Tænkepauser: Viden til hverdagen 47, Aarhus: Aarhus Universitetsforlag, 2017.
- [9] Vihart, *Doodling in Math: Spirals, Fibonacci, and Being a Plant*, YouTube, Part 1: <https://youtu.be/ahXIMUkSXX0>, Part 2: http://youtu.be/lOIP_Z_-0Hs, Part 3: <http://youtu.be/14-NdQwKz9w>, 2011–2012, last accessed on 24-05-2020.
- [10] Fibonacci numbers, *The On-Line Encyclopedia of Integer Sequences*, <https://oeis.org/A000045>, last accessed on 30-05-2020.