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Solutions 1601–1610

Q1601 Find conditions on a, b, c, d such that the curve

$$y = f(x) = x^4 + ax^3 + bx^2 + cx + d$$

has a double tangent.

SOLUTION Following the procedure in Problem 1591 (see solution last issue), we need to write $f(x) - (mx + n)$ as the square of a quadratic polynomial:

$$x^4 + ax^3 + bx^2 + (c - m)x + (d - n) = (x^2 + px + q)^2 .$$

Expanding and equating coefficients, this is true if and only if

$$2p = a\,, \quad 2q + p^2 = b\,, \quad 2pq = c - m\,, \quad q^2 = d - n\,.$$

If a, b, c, and d are given, then we can solve these equations one by one to find unique values for p, q, m and n. The curve $y = f(x)$ then has a double tangent if and only if the quadratic polynomial has two real roots: the condition for this is

$$\begin{aligned} p^2 - 4q > 0 \quad &\Leftrightarrow \quad p^2 - 2(b - p^2) > 0 \\ &\Leftrightarrow \quad 2b < 3p^2 \\ &\Leftrightarrow \quad b < \frac{3a^2}{8}\,. \end{aligned}$$

Q1602 A jar contains r red marbles and b blue marbles. If two of them are picked at random without replacement, then the probability that both the marbles drawn are red is 0.51, while the probability that both are blue is 0.07. Find r and b.

SOLUTION To save writing, let $p = 0.51$ and $q = 0.07$. Then the given probabilities are

$$\Big(\frac{r}{r+b}\Big)\Big(\frac{r-1}{r+b-1}\Big) = p \quad \text{and} \quad \Big(\frac{b}{r+b}\Big)\Big(\frac{b-1}{r+b-1}\Big) = q\,.$$

Multiplying out the denominators,

$$r(r-1) = p(r+b)(r+b-1) \ \text{ and } \ b(b-1) = q(r+b)(r+b-1)\,. \tag{1}$$

Now subtract these two equations and factorise the left hand side:

$$(r-b)(r+b-1) = (p-q)(r+b)(r+b-1)\,.$$

Clearly $r + b - 1$ is not zero, so we can cancel it, and it is then easy to rearrange and obtain

$$(1 + p - q)b = (1 - p + q)r\,. \tag{2}$$

Now equation (1) gives $q(1+p-q)^2r(r-1) = p(1+p-q)^2b(b-1)$, and substituting (2) into this yields

$$q(1+p-q)^2r(r-1) = p(1-p+q)r((1-p+q)r - (1+p-q)) .$$

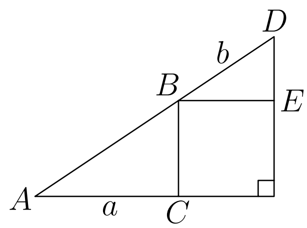
Once again r cancels, and then we have a simple (in principle) linear equation to solve. If you do the algebra very carefully, then you should find that

$$r = \frac{q(1+p-q)^2 - p(1-p+q)(1+p-q)}{q(1+p-q)^2 - p(1-p+q)^2} .$$

Finally, substituting $p = 0.51$ and $q = 0.07$ gives

$$r = 18 \quad \text{and} \quad b = \frac{1-p+q}{1+p-q}\, r = 7 .$$

Q1603 A right–angled triangle has a square drawn in the corner as shown. Find the area of the square in terms of the lengths $a = AC$ and $b = BD$.



SOLUTION Let the side of the square be x. We note that $\triangle ABC$ and $\triangle BDE$ are similar; using Pythagoras' Theorem too, we have

$$\frac{x}{a} = \frac{\sqrt{b^2 - x^2}}{x} .$$

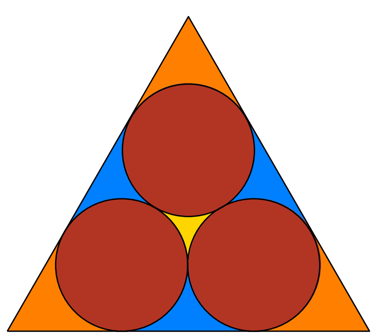
The required area is $S = x^2$, and the above can be written as

$$S = a\sqrt{b^2 - S} \quad \Leftrightarrow \quad S^2 = a^2(b^2 - S) .$$

By solving a quadratic equation and rejecting the negative root, we obtain the area

$$S = \frac{-a^2 + \sqrt{a^4 + 4a^2b^2}}{2} .$$

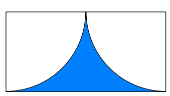
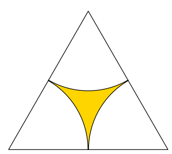
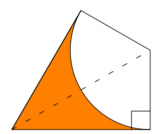
Q1604 A peculiar dartboard with a peculiar scoring system consists of an equilateral triangle with three inscribed circles of equal radius within.



The brown area is worth x points, the orange area is worth 3 points, the blue area is worth 9 points, and the gold area is worth 15 points. Assuming that a "hit" on this board has an equal chance of landing anywhere, what must the value of x be if the expected value of a "hit" is $6\sqrt{3} - 3$ points?

SOLUTION Only the relative sizes of the regions matter, not their absolute sizes. So we may assume that the circles have radius $r = 1$. We then calculate the areas of various sections.

- Each of the brown circles has area π.
- Each blue region can be seen as a 1 by 2 rectangle with two quarter–circles removed, and therefore has area $2 - \frac{\pi}{2}$.
- The gold region is an equilateral triangle with side 2 (joining the centres of the three circles), with three sixths of a circle removed, and has area $\sqrt{3} - \frac{\pi}{2}$.
- Each orange region is a kite which can be decomposed into two right–angled triangles, with one third of a circle removed. It has area $\sqrt{3} - \frac{\pi}{3}$.
- The whole dartboard is an equilateral triangle with side $2 + 2\sqrt{3}$ and therefore has area $\sqrt{3}(1 + \sqrt{3})^2 = 6 + 4\sqrt{3}$.

It would now be sensible to **check our working** by confirming that the total area of three brown regions, three blue, one gold and three orange is equal to the area of the whole dartboard.

The expected value of a hit is obtained by adding up the area of each region times its points value, and then dividing by the total area. Remembering that there are three regions of each colour except gold, this gives

$$\frac{3\pi x + 27(2 - \frac{\pi}{2}) + 15(\sqrt{3} - \frac{\pi}{2}) + 9(\sqrt{3} - \frac{\pi}{3})}{6 + 4\sqrt{3}}.$$

We know that this is equal to $6\sqrt{3} - 3$. Multiplying out the denominator and simplifying the numerator gives

$$3\pi x - 24\pi + 54 + 24\sqrt{3} = 54 + 24\sqrt{3}$$

and so $x = 8$.

Q1605 Calculate the constant term when the expression

$$\left(1 + x + \frac{1}{x}\right)^{10}$$

is expanded and like terms collected.

SOLUTION We wish to thank Arya R. Kondur, Monta Vista High School, California, USA, for submitting the following solution.

The expansion of the given trinomial can be done using the trinomial expansion formula,

$$(a+b+c)^n = \sum_{i+j+k=n} \binom{n}{i,j,k} a^i b^j c^k\,,$$

where n is a non–negative integer and the summation is taken over all combinations of non–negative integers i, j, k such that $i+j+k=n$. The trinomial coefficient is

$$\binom{n}{i,j,k} = \frac{n!}{i!\,j!\,k!}\,.$$

For our purposes, we have $a = 1$, $b = x$, $c = \frac{1}{x}$ and $n = 10$. Note that the term $a^i b^j c^k$ can be rewritten as

$$(1)^i (x)^j \left(\frac{1}{x}\right)^k.$$

This can be simplified to x^{j-k}. Notice that this term is a constant if, and only if, $j = k$. As such, the general term $\binom{n}{i,j,k} a^i b^j c^k$ is only constant if $j = k$. Since $i+j+k = 10$ and $j = k$, we have $i + 2j = 10$. We also have the condition that $i \geq 0$. Thus, we have $0 \leq j \leq 5$. It follows that the relevant triplets (i, j, k) used in the trinomial coefficients will be $(10, 0, 0)$, $(8, 1, 1)$, $(6, 2, 2)$, $(4, 3, 3)$, $(2, 4, 4)$ and $(0, 5, 5)$.

Note that the constant term can be expressed as a summation of only trinomial coefficients since there will be no variable part. Using the aforementioned triplets, the sum of the trinomial coefficients can be written as

$$\binom{n}{10,0,0} + \binom{n}{8,1,1} + \binom{n}{6,2,2} + \binom{n}{4,3,3} + \binom{n}{2,4,4} + \binom{n}{0,5,5}$$

where $n = 10$. Thus, we write

$$\binom{10}{10,0,0} + \binom{10}{8,1,1} + \binom{10}{6,2,2} + \binom{10}{4,3,3} + \binom{10}{2,4,4} + \binom{10}{0,5,5}$$

which can then be written as

$$\frac{10!}{10!\,0!\,0!} + \frac{10!}{8!\,1!\,1!} + \frac{10!}{6!\,2!\,2!} + \frac{10!}{4!\,3!\,3!} + \frac{10!}{2!\,4!\,4!} + \frac{10!}{0!\,5!\,5!}\,.$$

Performing the appropriate calculations yields $1 + 90 + 1260 + 4200 + 3150 + 252$. Thus, we conclude that the constant term in the trinomial expansion is 8953.

Alternative solution, for readers who have not met trinomial coefficients. Imagine the power written out as a product of 10 factors,

$$\left(1+x+\frac{1}{x}\right)\left(1+x+\frac{1}{x}\right)\cdots\left(1+x+\frac{1}{x}\right).$$

If we multiply out and *do not* collect terms, every product will have coefficient 1. To obtain a constant product we must choose m of the above factors to supply an x and m to supply a $1/x$; the remainder must supply a 1. Thus, we choose m of the 10 factors and then m of the remaining $10-m$; the number of ways of doing this is

$$\binom{10}{m}\binom{10-m}{m}.$$

In all this, m can be any integer from 0 to 5. So the number of constant products, and hence the total constant term when like terms are collected, is

$$\begin{aligned}\binom{10}{0}\binom{10}{0}+\binom{10}{1}\binom{9}{1}+\cdots+\binom{10}{5}\binom{5}{5}\qquad&\\ =1+90+1260+4200+3150+252&\\ =8953\,.&\end{aligned}$$

NOW TRY Problem 1611.

Q1606 In the following 4×4 array of numbers,

3	3	3	2
1	2	2	3
3	2	2	3
1	1	2	1

find a path which includes every square and does not visit any square more than once. You may start in any square you wish and finish in any square you wish; from a square containing the number n you must move to a square which is n squares away (either horizontally, vertically or diagonally).

SOLUTION Label the squares a to p as shown.

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

We begin by listing all possible moves:

$$ad,\ am,\ ap;\quad bn;\quad co;\quad db,\ dj,\ dl;\quad ea,\ eb,\ ef,\ ei,\ ej;$$

$$fh,\ fn,\ fp; \quad ge,\ gm,\ go; \quad he; \quad il; \quad jb,\ jd,\ jl; \quad ka,\ kc,\ ki;$$
$$li; \quad mi,\ mj,\ mn; \quad ni,\ nj,\ nk,\ nm,\ no; \quad oe,\ og,\ om; \quad pk,\ pl,\ po.$$

For convenience, here are the same moves, but ordered according to the destination rather than the origin:

$$ea,\ ka; \quad db,\ eb,\ jb; \quad kc; \quad ad,\ jd; \quad ge,\ he,\ oe; \quad ef; \quad og; \quad fh;$$
$$ei,\ ki,\ li,\ mi,\ ni; \quad dj,\ ej,\ mj,\ nj; \quad nk,\ pk; \quad dl,\ il,\ jl,\ pl;$$
$$am,\ gm,\ nm,\ om; \quad bn,\ fn,\ mn; \quad co,\ go,\ no,\ po; \quad ap,\ fp.$$

We need to list the letters a to p in a sequence such that every pair of adjacent letters in the sequence comes from the above pairs. First note that i can go only to l, and l can go only to i; so one of these letters must be the last in the sequence. For a start, we try making i the last. This means that b, c, h and l are not the last letters in the sequence; they must each be followed by another letter; and in each case there is only one choice. So our sequence must contain the following, where we use] to denote the end of the sequence:

$$bn \qquad co \qquad he \qquad li\,]$$

Now g must be followed by e, m or o; but e, o are impossible as they are already preceded by other letters; so we must have gm; and by similar arguments mj and nk and og and pl and db and jd. The situation now is

$$cogmjdbnk \qquad he \qquad pli\,]$$

The only letter which could precede c is k, which is impossible; so c must be the first letter in the sequence; then k must be followed by a and a by p and we have

$$[\,cogmjdbnkapli\,] \qquad he$$

We now have a complete sequence from start to finish; but it does not contain all sixteen letters; so we have, finally, ruled out the possibility that i is the last letter.

Therefore, l is the last. By similar arguments to the above, we must have bn and co and he and il and gm and og and pk, giving

$$pk \qquad cogm \qquad bn \qquad he \qquad il\,] \tag{$*$}$$

At this stage we can deduce that the first letter is either f or h. For if not then each must have a predecessor; there is only one choice in each case; and we obtain the string $hefh$, which is impossible as it contains h twice. If h is the first letter, then f must be preceded by e, so that we get

$$[\,hef \qquad pk \qquad cogm \qquad bn \qquad il\,]$$

This forces us to choose fp and kc (because no letter but k precedes c) and mj (because mi would make the sequence finish too soon) and ad and jb, so that we have

$$[\,hefpkcogmjbn \qquad ad \qquad il\,]$$

and now there is no available letter to follow d. So we return to $(*)$ knowing that f is the first letter:

$$[\,f \qquad pk \qquad cogm \qquad bn \qquad he \qquad il\,]$$

Now h must be preceded by f, and c must be preceded by k, and p by a, and a by e, and i by n. We have

$$[\,fheapkcogm \qquad bnil\,]$$

We have the beginning and end of the sequence, with the letters d and j to go between them; and it is easy to check that jd works while dj doesn't. So the only possible solution is

$$[\,fheapkcogmjdbnil\,]\,.$$

Q1607 How many triples (x, y, z) of positive integers satisfy the conditions

$$x < y < z \quad \text{and} \quad x + xy + xyz = 2019\,?$$

SOLUTION First notice that

$$x^3 < xyz < x + xy + xyz = 2019\,,$$

and so $x \leq 12$. The given equation can be written

$$x(1 + y(1 + z)) = 2019 = 3 \times 673\,,$$

and 673 is prime; since x is a factor of 2019 we have $x = 1$ or $x = 3$.

- If $x = 1$, then $y(1 + z) = 2018 = 2 \times 1009$, and 1009 is prime; so the only solution is $y = 2$ and $z = 1009$.
- If $x = 3$, then $y(1 + z) = 672 = 2^5 \times 3 \times 7$. Factorising this number gives 24 possibilities for y and $1 + z$; but half of these have $y > z$ and must be discarded; while $y = 1, 2, 3$ are not greater than x and must also be discarded. So there are 9 solutions.

Therefore there are 10 possible solutions altogether.

Q1608 Define a decimal x by concatenating the digits in the sequence of perfect squares,

$$x = 0 \cdot 149162536496481100121 \cdots\,.$$

Prove that x is not a rational number.

SOLUTION Suppose that x is rational. Then the decimal which defines x must be eventually recurring (that is, recurring after a possible initial non–recurring part). Suppose that the recurring part has length m; choose an integer n such that $n \geq m$ and the digits of $(10^n)^2$ occur within the recurring part of the decimal. Since

$$(10^n)^2 = 1\overbrace{00\cdots00}^{2n\ \text{zeros}}\,,$$

the recurring part contains m consecutive zeros; that is, the recurring digits are only zeros; so the decimal is zero from some point onwards. But this is clearly not true. Hence, it is impossible that x is rational.

Q1609 Simplify the sum

$$\sec 0 \sec 1 + \sec 1 \sec 2 + \sec 2 \sec 3 + \cdots + \sec(n-1) \sec n \,.$$

SOLUTION First note that all of the $\sec$ terms in the sum are defined, since an integer can never be an odd multiple of $\frac{\pi}{2}$. For any k we have

$$\begin{aligned}
\sec(k-1)\sec k &= \frac{1}{\cos(k-1)\cos k} \\
&= \frac{\sin(k-(k-1))}{\sin 1 \cos(k-1)\cos k} \\
&= \frac{1}{\sin 1}\,\frac{\sin k \cos(k-1) - \cos k \sin(k-1)}{\cos(k-1)\cos k} \\
&= \frac{1}{\sin 1}\big(\tan k - \tan(k-1)\big)\,.
\end{aligned}$$

Therefore the given sum is

$$\frac{1}{\sin 1}\big((\tan 1 - \tan 0) + (\tan 2 - \tan 1) + (\tan 3 - \tan 2) + \cdots + (\tan n - \tan(n-1))\big) = \frac{\tan n}{\sin 1}\,.$$

Q1610 Suppose that we have a bag containing n different objects. We wish to choose a collection of objects A from S: the collection A may contain all the objects in S, or none, or anywhere in between. Then we want to choose two collections B and C from the objects in A: these, again, may contain any number of objects from A, except that there must be at least one object which is in both B and C. In how many ways can the choice of the three collections A, B, and C be made?

SOLUTION There are five possible destinations for any object in our original bag: in B but not C; in C but not B; in both B and C; in A but neither B nor C; not in A. So the number of choices is 5^n. However, this includes options where no object was chosen in both B and C, which violates the conditions of the problem. So we must count how many such cases there are and subtract them. To do this, simply note that we now have only four possible destinations for each object ("both B and C" is ruled out), giving 4^n choices.

Therefore, the number of choices that we actually want to count is $5^n - 4^n$.