Parabola Volume 56, Issue 2 (2020)

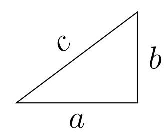
Quasi-Pythagorean Triads

Stephen Bewlay[1] and **Long Yin Felicina Chau**[2]

1 Introduction

Triangles – like other geometric plane and 3-dimensional objects - have a unique geometry of their own and obey their own set of unique rules. Triangles are very "three". They are saturated with "three-ness"! They have 3 sides. They have 3 interior angles. They have 3 exterior angles. You can generate them by simply drawing 3 straight lines.

One of the most important classes of triangles is the class of right-angled triangles, which obey the Pythagorean equation:



$$a^2 + b^2 = c^2 \tag{1}$$

where a and b are the lengths of the orthogonal (i.e., perpendicular) sides and c is the length of the "slopey" side (the hypotenuse). Even here, the three-ness of these triangles permeates the entire equation – the formula involves exactly 3 terms, tied together in a simple way!

For simplicity, when people are constructing things out of triangles, it would often be convenient for the lengths of all 3 sides of the triangle to be simple integer values. We can arbitrarily fix two of the sides to be anything we like, so we can easily make these both integers. BUT, the 3rd side for a right-angled is non-negotiable! It is slavishly dictated by the Pythagorean relationship. If, for example, we were hoping to have a triangle with a, b and c values of (say) 1, 2 and 3 respectively, or 1, 2 and 4 or 1, 3 and 4 or 1, 3 and 5, then we will be disappointed.

The a and b values 1 and 2 belong to a c value of $\sqrt{5}$, which is about 2.236 and is definitely not an integer. The a and b values 1 and 3 belong to a c value of $\sqrt{10}$, which is about 3.162 and is, once again, not an integer.

[1]Stephen Bewlay is Intellectual Property advisor at Eco-Sight Corp (eco_sight@yahoo.com).
[2]Long Yin Felicina Chau is a design graduate with a passion for Maths and IT, currently studying in the School of Mathematics and Applied Statistics at the University of Wollongong (fchau1810@gmail.com).

2 History

Somebody - maybe multiple somebodies in different places and times – first discovered the relationship which is traditionally linked to Pythagoras, who lived from 572 to 497 BCE, but the "Pythagorean" relationship (1) was most likely discovered before Pythagoras. For example, in India, some "Sulba-sutra" writings of Vedic scholars contain mathematical concepts and are dated between about 800 and 200 BCE. The four major, and most mathematically significant of these, were authored by Baudhayana, Manava, Apastamba and Katyayana. If the 200 BCE date for the Baudhayana Sulba-sutra is correct, then it means the Indians hit on the idea a bit later than Pythagoras, but if the 800 BCE date is correct, then it means the Indians easily beat Pythagoras to the goal! It's very interesting to read how the Baudhayana Sulba-sutra states the Pythagorean relationship: algebra – and a lot of other mathematical symbolism - hadn't been invented then, and the statement is in the form of words or – more accurately – a free-form POEM! In English, it translates as:

> "The rope stretched along the length of the diagonal of a rectangle makes an area which the vertical and horizontal sides make together".

In a more mathematical form, it's saying:

$$d^2 = v^2 + h^2 \,. \tag{2}$$

Since our choice of pronumerals is entirely arbitrary and for convenience, this is clearly identical to Equation (1) above!

The Baudhayana Sulba-sutra gives examples of Pythagorean triples given as the side-lengths of right angled triangles. Different references to this work vary in their claims. Some authors claim that 5 different examples of Pythagorean triads are listed in the Sulba-sutra; others mention only 4. The 4 that seem to be agreed on by most of authors/reviewers are:

$$\begin{array}{rrrr} & 5, & 12, & 13\,; \\ & 8, & 15, & 17\,; \\ & 12, & 16, & 20\,; \\ \text{and} & 12, & 35, & 37 \end{array}$$

(see the Ian Pearce reference in Reference list at end).

The oldest known Chinese mathematical work is the "Zhoubi Suanjing" (also spelt Chou Pei Suan Ching or Chou Pi Suan Ching), composed sometime between 100 BC and 100 AD. The author of the compilation is unknown, and scholars agree that most of the contents originated from a much earlier period. The "Zhoubi" was divided into two sections, the first dealing with mathematics and the second part with astronomy. The mathematics in section one was needed first before the reader could understand the explanations of astronomy in the second section! Section One included a discussion of right-triangle theory. This is known as the "Gou-gu Theorem" or the "Shang Gao theorem" and is mentioned in other ancient Chinese literature too. (What is commonly referred to as "Pythagoras' Theorem" in the Western World, was usually called the “Gou-gu Theorem” or ”Shang Gao Theorem” throughout China.) Interestingly, the teaching about this theorem is written as a dialogue between a teacher Chen Zi and a student Rong Fang. (Different cultures, different styles. Magnificent variety!) Unfortunately, Chen Zi and Rong Fang don’t show up anywhere else in ancient Chinese writings, so we have no more clues about who they were!

The Zhoubi records that the Emperor Yu or “Yu the Great” in 2070 BCE was able to rule the country because he appreciated and applied the “Gougu Theorem”. He is credited for controlling a huge flood by engineering dredges and alternative flood channels to control the water. In pictures, he is often seen holding a set-square. This is evidence the Chinese understood and were applying the properties of right-angle triangles as early as 2070 BCE. There is also very elegant Chinese geometric drawing of a proof of Pythagoras’ Theorem in the Zhoubi Suanjing (see Figure 1), but the earliest records of this work are from sometime between 100 BC and 100 AD and it is hard to attribute a definite date to when the concept was first understood and taught.

During the Han Dynasty (202 BCE to 220 AD), definite sets of Pythagorean triples were included in “The Nine Chapters on the Mathematical Art”, but this date is a little after Pythagoras’ time.

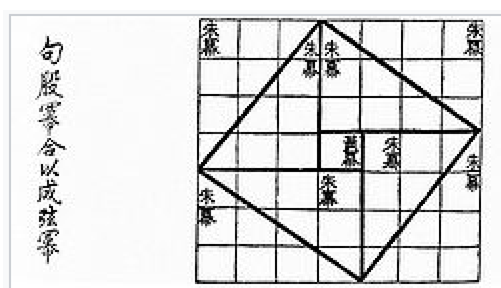


Figure 1: The Babylonian “Plimpton 322” clay tablet and a Chinese proof of Pythagoras’ Theorem

If you're interested in ancient knowledge about Pythagorean triangles, then try looking for pictures and information about Plimpton 322, shown in Figure 1 above. It is the Babylonian cuneiform clay tablet with the world's oldest-known list of Pythagorean triples, from the Old Babylonian era, between the 19th and 16th centuries B.C.E. See also the References given at the end, and especially [9].

3 Pythagorean Triads

Table 1 below shows some of the simplest number combinations for a, b and c which can be combined to construct a right-angles triangle where all 3 sides have an integer length (i.e., where a, b and c are in a Pythagorean relationship and thereby a TRUE Pythagorean triad).

Table 1: Traditional Known Pythagorean Triads

Traditional Known Pythagorean Triads				**Traditional Known Pythagorean Triads**				**Traditional Known Pythagorean Triads**			
a	**b**	**c**		**a**	**b**	**c**		**a**	**b**	**c**	
				6	**8**	**10**	2×(3, 4, 5)				
								9	**12**	**15**	3×(3, 4, 5)
3	**4**	**5**									
								9	**40**	**41**	
				7	**24**	**25**		**10**	**24**	**26**	2×(5, 12, 13)
								12	**16**	**20**	4×(3, 4, 5)
								14	**48**	**50**	2×(7, 24, 25)
								15	**20**	**25**	5×(3, 4, 5)
								16	**30**	**34**	2×(8, 15, 17)
								20	**21**	**29**	
								21	**72**	**75**	3×(7, 24, 25)
								24	**32**	**40**	8×(3, 4, 5)
				8	**15**	**17**		**40**	**42**	**58**	2×(20, 21, 29)
								80	**84**	**116**	4×(20, 21, 29)
								80	**192**	**208**	16×(5, 12, 13)
5	**12**	**13**						**100**	**105**	**145**	5×(20, 21, 29)
								120	**209**	**241**	

As you can see, there are a lot of blank spaces in the table. That's because we left a lot of space for combinations which people might hope for like

$$\begin{array}{llll} 1, & 2, & 4 & \text{or} \\ 1, & 3, & 4 & \text{or} \\ 1, & 3, & 5 & \text{or} \\ 2, & 2, & 3 & \text{or} \\ 2, & 3, & 4 & \text{or} \\ 2, & 4, & 8 & \text{etc.} \end{array}$$

In fact, NONE of the combinations are Pythagorean until we get to the number 3, and even then NOTHING gives a right-angled triangle with all-integer side-lengths until we get to $a = 3$ and $b = 4$ which has a hypotenuse of 5. Hence, 3, 4, 5 is the simplest possible "Pythagorean Triad" or "Pythagorean Triple" of integers.

After 3, 4, 5, the next triple of integers that fits a right-angled triangle is 5, 12, 13. After that, we can try an a value of 6 and get 6, 8, 10 by multiplying all the 3, 4, 5 triangle side-lengths by 2. After that, the next new Pythagorean triple is 7, 24, 25, and then there's a big gap until we arrive at 8, 15, 17. After this, the allowable combinations start to get quite sparse, until we arrive at 9, 12, 15 (which is just 3×(3, 4, 5)). We have another entirely new combination at 9, 40, 41, then a few combinations turn up from multiples of the earlier, smaller integers. The next entirely new combination is 20, 21, 29, followed by a whole bunch of bigger multiples of smaller earlier triads, and nothing really new turns up until you reach the combination $a, b, c = 120, 209, 241$, respectively. There are still Pythagorean triads turning up beyond that, but they get a bit harder to find. That's because both the a and b numbers you are trying to combine are both getting bigger and bigger, and – once you've chosen a - you have to check a bigger and bigger range of b values.

4 Filling in the blanks

Humans can be suspicious of blanks. They look untidy. Was something accidentally missed?!! You may have thought that it seemed weird that school didn't teach you about Pythagorean triads with $a = 4$ or $a = 11$. And why are there so many other large blank regions in Table 1?? What if previous mathematicians accidentally missed some combinations that were actually Pythagorean?! You may have played around with combinations like "7, 14, . . ." to see if you could find an overlooked Pythagorean triple, but unfortunately 7 and 14 give a c value of 15.65248, which you can't pretend is an integer. Likewise, "10, 20, . . ." belongs with $c = 22.36$ which can't be treated as an integer without blatantly truncating it and introducing a severe round-off error.

BUT, what about "8, 31, . . ."??? Its corresponding c value is 32.016. ".016" sounds pretty darn small. What if we just ignored it and pretended that the c value was "32"?! Calling "32.016" "32" introduces an error of only $0.016/32.016 \times 100\%$, which is just 0.049%. Many people doing real-life applications with triangles wouldn't even notice a 0.049% error. They would never notice and never care! Maybe their work only needs an accuracy of $\sim \pm 0.5\%$! Gee! We're onto something here. For simple everyday purposes, they could call 8, 31, 32 a "Pythagorean Triad"! It's accurate to within 0.049%. To forestall complaints, placate the Pure Mathematicians and stop them shouting us down in disgust (or lynching us), we suggest a cunning terminology shift whereby we call this wonderful new beast a "Quasi-Pythagorean Triad"! Are there MORE of them out there?? This is a whole new frontier!

Table 2: Quasi-Pythagorean Triads accurate to 0.1% or better

Pythag. & Quasi - Triads				Pythag. & Quasi - Triads				Pythag. & Quasi - Triads			
a	b	"c"	c Actual	a	b	"c"	c Actual	a	b	"c"	c Actual
1	22	22	22.02								
1	23	23	23.02	**6**	**8**	**10**	2×(3, 4, 5)				
1	38	38	38.01					**9**	**12**	**15**	3×(3, 4, 5)
2	43	43	43.05								
2	45	45	45.04								
3	**4**	**5**						9	39	40	40.0250
								9	**40**	**41**	
				7	**24**	**25**		**10**	**24**	**26**	2×(5, 12, 13)
								11	57	58	58.0517
								12	**16**	**20**	4×(3, 4, 5)
3	67	67	67.07					**14**	**48**	**50**	2×(7, 24, 25)
								15	**20**	**25**	5×(3, 4, 5)
								16	**30**	**34**	2×(8, 15, 17)
								20	**21**	**29**	
								20	25	32	32.0156
								21	**72**	**75**	3×(7, 24, 25)
4	87	87	87.09								
4	89	89	89.09					**24**	**32**	**40**	8×(3, 4, 5)
4	90	90	90.09					29	29	41	41.01
				8	**15**	**17**		**40**	**42**	**58**	2×(20, 21, 29)
				8	31	32	32.02				
				8	32	33	32.98	75	95	121	121.0372
				8	33	34	33.75	**80**	**84**	**116**	4×(20, 21, 29)
								80	**192**	**208**	16×(5, 12, 13)
								100	**105**	**145**	5×(20, 21, 29)
5	**12**	**13**									
								120	**209**	**241**	
								198	198	280	280.01
								300	360	469	468.6150
								300	380	484	484.1487
								394	394	557	557.20

5 Quasi-Pythagorean Triads

How about a triangle with side lengths "6, 16, ..."? It turns out that the c value for this combination is 17.088. That's pretty close to 17. Only 0.515% different. So, you could say that 6, 16, 17 is a Quasi-Pythagorean Triad to an accuracy of much better than $\pm 1\%$ (almost 0.5%). We hunted down combinations of a, b and c (up to 3-digit numbers) which were close to Pythagorean ("Quasi-Pythagorean"). Even if an accuracy of $\pm 0.10\%$ is demanded, we found quite a few combinations of integers that fulfilled the conditions. Table 2 above shows some typical results.

Table 2 is by no means exhaustive or complete. We merely tried to cover a span of values that gives the reader a practical and realistic view of how many combinations turn up, while keeping the size no bigger than a normal page or screen. Note that, very pleasingly, there ARE a few Quasi-Pythagorean Triples that start with $a = 4$, and there's a few combinations starting with $a = 8$ and also $a = 11$! (that's an exclamation mark, not a factorial sign!).

Table 3: Quasi-Pythagorean Triads accurate to 0.5% or better

Pythag. & Quasi - Triads				Pythag. & Quasi - Triads				Pythag. & Quasi - Triads			
a	b	"c"	c Actual	a	b	"c"	c Actual	a	b	"c"	c Actual
1	22	22	22.02								
1	23	23	23.02	6	8	10	2×(3, 4, 5)				
1	38	38	38.01					9	12	15	3×(3, 4, 5)
				6	16	17	17.088				
2	38	38	38.05								
2	43	43	43.05	6	24	25	24.739	9	20	22	21.93
2	45	45	45.04								
3	4	5						9	39	40	40.0250
				7	11	13	13.038	9	40	41	
								10	24	26	2×(5, 12, 13)
				7	22	23	23.087	11	19	22	21.95
				7	24	25	25.000	11	57	58	58.0517
				7	25	26	25.752	12	16	20	4×(3, 4, 5)
3	30	30	30.15	7	26	27	26.926	13	15	20	19.8494
3	67	67	67.07	7	27	28	27.893	14	48	50	2×(7, 24, 25)
				7	28	29	28.862				
								15	20	25	5×(3, 4, 5)
								16	30	34	2×(8, 15, 17)
								20	21	29	
4	30	30	30.265					20	25	32	32.0156
								21	72	75	3×(7, 24, 25)
4	87	87	87.09								
4	89	89	89.09					24	32	40	8×(3, 4, 5)
4	90	90	90.09					29	29	41	41.01
				8	9	12	12.042				
								40	42	58	2×(20, 21, 29)
				8	15	17					
								75	95	121	121.0372
				8	29	30	30.08	80	84	116	4×(20, 21, 29)
				8	30	31	31.05	80	192	208	16×(5, 12, 13)
				8	31	32	32.02	100	105	145	5×(20, 21, 29)
5	12	13		8	32	33	32.98				
5	13	14	13.928	8	33	34	33.75	120	209	241	
				8	34	35	34.93	198	198	280	280.01
								300	360	469	468.6150
								300	380	484	484.1487
								394	394	557	557.20

6 "But I'm in the Landscape Gardening Business!"

Some people using triangles don't need an accuracy of 0.1%. What about the landscape gardener who occasionally designs triangular garden beds? Sometimes the garden borders might be just lines of local rocks dotted along in a (reasonably) straight line on the side of a hill of variable slope. When doing the layout, it would be convenient to have all the dimensions in integer values so that they're very easy to measure with a measuring tape. But maybe they don't need to be Pythagorean – just Quasi-Pythagorean. Maybe they need an accuracy of $\pm 1\%$, perhaps $\pm 2\%$, or even $\pm 5\%$?!

To see what this means for the available range of a, b, c values, we constructed Tables 3–6, with each table covering ever-decreasing levels of accuracy. Even at the $\pm 1\%$

accuracy level (Table 4), there are MANY wonderful new quasi-Pythagorean "right-angled" (nearly!) triangle combinations available, e.g.:

$$\begin{array}{lll} 5, & 11, & 12 \\ 5, & 13, & 14 \\ 7, & 11, & 13 \quad \text{(your lucky place to shop!)} \\ 9, & 20, & 22 \\ 9, & 39, & 40\,. \end{array}$$

Table 4: Quasi-Pythagorean Triads accurate to 1% or better

Pythagorean & Quasi - Triads				Pythag. & Quasi - Triads				Pythag. & Quasi - Triads			
a	**b**	**"c"**	**c Actual**	**a**	**b**	**"c"**	**c Actual**	**a**	**b**	**"c"**	**c Actual**
1	22	22	22.02								
1	23	23	23.02	**6**	**8**	**10**	2×(3, 4, 5)				
1	38	38	38.01	6	15	16	16.155	**9**	**12**	**15**	3×(3, 4, 5)
				6	16	17	17.088				
2	38	38	38.05	6	24	25	24.74				
2	43	43	43.05	7	7	10	9.899 0.010050506	9	20	22	21.93
2	45	45	45.04	7	11	13	13.038	9	22	24	23.77
3	**4**	**5**		7	12	14	13.892	9	39	40	40.0250
								9	**40**	**41**	
				7	20	21	21.190	10	11	15	14.8661
				7	21	22	22.136	**10**	**24**	**26**	2×(5, 12, 13)
				7	22	23	23.087	11	19	22	21.95
				7	**24**	**25**	25.000	11	57	58	58.0517
				7	25	26	25.752	12	16	20	4×(3, 4, 5)
3	30	30	30.15	7	26	27	26.926	13	15	20	19.8494
3	67	67	67.07	7	27	28	27.893	**14**	**48**	**50**	2×(7, 24, 25)
				7	28	29	28.862				
				7	34	35	34.713	**15**	**20**	**25**	5×(3, 4, 5)
				7	35	36	35.693				
				7	36	37	36.674	**16**	**30**	**34**	2×(8, 15, 17)
				7	39	40	39.623	20	20	28	28.28
				7	40	41	40.608	**20**	**21**	**29**	
4	30	30	30.265	7	42	43	42.579	20	25	32	32.0156
				7	43	44	43.566	**21**	**72**	**75**	3×(7, 24, 25)
4	87	87	87.09	7	44	45	44.553				
4	89	89	89.09	7	46	47	46.530	**24**	**32**	**40**	8×(3, 4, 5)
4	90	90	90.09	7	50	50	50.488	29	29	41	41.01
				8	9	12	12.042	30	35	46	46.0977
								40	**42**	**58**	2×(20, 21, 29)
				8	**15**	**17**		41	70	81	81.12
								75	95	121	121.0372
				8	29	30	30.08	**80**	**84**	**116**	4×(20, 21, 29)
				8	30	31	31.05	**80**	**192**	**208**	16×(5, 12, 13)
5	11	12	12.083	8	31	32	32.02	**100**	**105**	**145**	5×(20, 21, 29)
5	**12**	**13**		8	32	33	32.98	100	110	149	148.6607
5	13	14	13.928	8	33	34	33.75	**120**	**209**	**241**	
				8	34	35	34.93	198	198	280	280.01
								300	360	469	468.6150
								300	380	484	484.1487
								394	394	557	557.20

7 The "Garden 4" Project

The Nuns of St. Gnosticus recently had an old out-building demolished and removed from their property, thus yielding an extra 20.1m ×20.1m of garden land (>404.01m^2 – something really aesthetic about that number - check the math!) After the stress of the recent Royal Commission, they hatched an exciting (and so-far top secret) plan to use the new area for extra garden space for gardening, animal habitat and quiet contemplation (The "Garden 4" project). This will extend their existing kitchen-garden area and include many more ornamental plants/flowers and shrubs as well. They've drawn up some wonderful plans featuring a network of walkways crossing in the centre with lots of benches installed along the paths for rest and meditation in the peaceful outdoors in between work. They're hoping for the Bishop to sign off on the plans at the diocesan meeting next month. Figure 2 shows the proposed "Garden 4" plans.

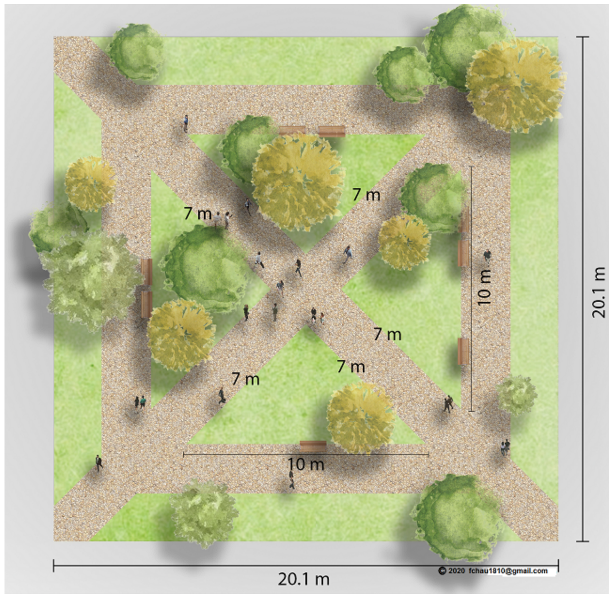


Figure 2: "Garden 4" Plan of St. Gnosticus

Table 5: Quasi-Pythagorean Triads accurate to 2% or better

Pythag. & Quasi - Triads				Pythag. & Quasi - Triads				Pythag. & Quasi - Triads			
a	b	"c"	c Actual	a	b	"c"	c Actual	a	b	"c"	c Actual
1	22	22	22.02								
1	23	23	23.02	**6**	**8**	**10**	2×(3, 4, 5)				
1	38	38	38.01	6	15	16	16.155	**9**	**12**	**15**	3×(3, 4, 5)
				6	16	17	17.088				
2	38	38	38.05	6	24	25	24.74				
2	43	43	43.05	7	7	10	9.899	9	20	22	21.93
2	45	45	45.04	7	11	13	13.038	9	22	24	23.77
3	**4**	**5**		7	12	14	13.892	9	39	40	40.0250
				7	13	15	14.765	**9**	**40**	**41**	
				7	20	21	21.190	10	11	15	14.8661
				7	21	22	22.136	**10**	**24**	**26**	2×(5, 12, 13)
				7	22	23	23.087	11	19	22	21.95
				7	**24**	**25**	25.000	11	57	58	58.0517
				7	25	26	25.752	**12**	**16**	**20**	4×(3, 4, 5)
3	30	30	30.15	7	26	27	26.926	13	15	20	19.8494
3	67	67	67.07	7	27	28	27.893	**14**	**48**	**50**	2×(7, 24, 25)
				7	28	29	28.862				
				7	34	35	34.713	**15**	**20**	**25**	5×(3, 4, 5)
4	7	8	8.062	7	35	36	35.693				
4	8	9	8.944	7	36	37	36.674	**16**	**30**	**34**	2×(8, 15, 17)
4	9	10	9.849	7	39	40	39.623	20	20	28	28.28
4	20	20	20.375	7	40	41	40.608	**20**	**21**	**29**	
4	30	30	30.265	7	42	43	42.579	20	25	32	32.0156
				7	43	44	43.566	**21**	**72**	**75**	3×(7, 24, 25)
4	87	87	87.09	7	44	45	44.553				
4	89	89	89.09	7	46	47	46.530	**24**	**32**	**40**	8×(3, 4, 5)
4	90	90	90.09	7	50	50	50.488	29	29	41	41.01
				8	9	12	12.042	30	35	46	46.0977
				8	10	13	12.806	**40**	**42**	**58**	2×(20, 21, 29)
				8	**15**	**17**		41	70	81	81.12
				8	19	21		75	95	121	121.0372
				8	29	30	30.08	**80**	**84**	**116**	4×(20, 21, 29)
5	10	11	11.180	8	30	31	31.05	**80**	**192**	**208**	16×(5, 12, 13)
5	11	12	12.083	8	31	32	32.02	**100**	**105**	**145**	5×(20, 21, 29)
5	**12**	**13**		8	32	33	32.98	100	110	149	148.6607
5	13	14	13.928	8	33	34	33.75	**120**	**209**	**241**	
				8	34	35	34.93	198	198	280	280.01
								300	360	469	468.6150
								300	380	484	484.1487
								394	394	557	557.20

Table 6: Quasi-Pythagorean Triads accurate to 5% or better

Pythag. & Quasi - Triads				Pythag. & Quasi - Triads				Pythag. & Quasi - Triads			
a	b	"c"	c Actual	a	b	"c"	c Actual	a	b	"c"	c Actual
1	22	22	22.02	6	7	9	9.220				
1	23	23	23.02	**6**	**8**	**10**	2×(3, 4, 5)	9	10	13	13.45
1	38	38	38.01	6	15	16	16.155	**9**	**12**	**15**	3×(3, 4, 5)
				6	16	17	17.088				
2	38	38	38.05	6	24	25	24.74				
2	43	43	43.05	7	7	10	9.899	9	20	22	21.93
2	45	45	45.04	7	11	13	13.038	9	22	24	23.77
3	**4**	**5**		7	12	14	13.892	9	39	40	40.0250
3	5	6	5.83	7	13	15	14.765	**9**	**40**	**41**	
3	6	7	6.708	7	20	21	21.190	10	11	15	14.8661
3	7	8	7.616	7	21	22	22.136	**10**	**24**	**26**	2×(5, 12, 13)
3	10	10	10.440	7	22	23	23.087	11	19	22	21.95
3	11	11	11.402	**7**	**24**	**25**	25.000	11	57	58	58.0517
3	12	12	12.369	7	25	26	25.752	**12**	**16**	**20**	4×(3, 4, 5)
3	30	30	30.15	7	26	27	26.926	13	15	20	19.8494
3	67	67	67.07	7	27	28	27.893	**14**	**48**	**50**	2×(7, 24, 25)
				7	28	29	28.862				
4	6	7	7.211	7	34	35	34.713	**15**	**20**	**25**	5×(3, 4, 5)
4	7	8	8.062	7	35	36	35.693				
4	8	9	8.944	7	36	37	36.674	**16**	**30**	**34**	2×(8, 15, 17)
4	9	10	9.849	7	39	40	39.623	20	20	28	28.28
4	20	20	20.375	7	40	41	40.608	**20**	**21**	**29**	
4	30	30	30.265	7	42	43	42.579	20	25	32	32.0156
				7	43	44	43.566	**21**	**72**	**75**	3×(7, 24, 25)
4	87	87	87.09	7	44	45	44.553				
4	89	89	89.09	7	46	47	46.530	**24**	**32**	**40**	3×(8, 4, 5)
4	90	90	90.09	7	50	50	50.488	29	29	41	41.01
				8	9	12	12.042	30	35	46	46.0977
5	5	7	7.07	8	10	13	12.806	**40**	**42**	**58**	2×(20, 21, 29)
5	6	8	7.810	**8**	**15**	**17**		41	70	81	81.12
5	7	9	8.602	8	19	21		75	95	121	121.0372
5	8	9	9.434	8	29	30	30.08	**80**	**84**	**180**	4×(20, 21, 29)
5	10	11	11.180	8	30	31	31.05	**80**	**192**	**208**	16×(5, 12, 13)
5	11	12	12.083	8	31	32	32.02	**100**	**105**	**145**	5×(20, 21, 29)
5	**12**	**13**		8	32	33	32.98	100	110	149	148.6607
5	13	14	13.928	8	33	34	33.75	**120**	**209**	**241**	
				8	34	35	34.93	198	198	280	280.01
								300	360	469	468.6150
								300	380	484	484.1487
								394	394	557	557.20

The Surveyor assured the Mother Superior that the external dimensions are exactly 20.1m×20.1m and the Nuns have a 10m tape-measure. The Nuns have planned lots of symbolism into the garden. They intend the perimetral garden strips to be 4 in number and to be 2m wide, and all the walking paths will also be 2m wide. There are 3 entrance/exit points, 10 benches and 11 path segments (the 12 apostles minus Judas I.) and – of course – the paths form a cross (naturally). The overall shape is square (like the footprint of the city of heaven) and Mother Superior Margueretta (who was a Math major at the Universite de Roma) suggested the central triangular garden beds each be 7, 7, 10 quasi-pythagorean triangles (having about $24.5m^2$ each) as shown. Checking all the math is left as an exercise for the reader!

8 A Magnificent Gleaming Tower

His Excellency the Crown Prince Ahmed bin Abdul bin Sulaiman of the nth generation (where n is a large-ish integer) has plans for a magnificent 200-storey-high skyscraper in his capital city. The orientation of this building has been skilfully chosen by His Excellency and his appointed award-winning Architect. The top of the building will be faceted with a large area of flat, gold-coated glass, but there is uncertainty at this stage whether the building should resemble any of the shapes in Figure 3. The Prince has tasked the Architect to search the scrolls of ancient wisdom to see if the sages taught a combination of a, b and c - all integers - which will give a θ angle of $77° \pm 0.1\%$. The total width w of the building in the relevant direction is 30m.

Figure 3: Proposed Geometries for the "Desert Beacon" Skyscraper Tower

If the target angle can be achieved, using integer (or quasi-integer values accurate to 0.10%), an impressive world-first will be achieved. The builders will also have a reasonably easy measurement job and the sun around 1:00 pm on the Prince's birthday (the 3rd day of Ramadan) will always be reflected by the facet section towards the exact region of desert dunes which were the traditional nomadic grazing lands of the Prince's great grandfather.

From our Table 2, there is a nice set of values 9, 39, 40 which can be doubled to produce 18m, 78m, 80m, yielding a θ value of $77.005°$, which is an angular accuracy of 0.065%, and the c value is accurate to 0.062%, well within the Prince's requested range. This will make the facet section of the building be approximately midway between the second and third configurations of Figure 3.

9 Conclusion

We have only just scratched the surface of this interesting topic. There are many more things which someone could do with Quasi-Pythagorean Triads/Triples and the triangles they describe. How about isosceles triangles morphing into right angled triangles, for example, or super-skinny triangles?! Feel free to feedback error-corrections and suggestions and why not put together an article about your findings for *Parabola*?!

References

[1] R. Knott, *Pythagorean Right-Angled Triangles*, (accessed 14/3/2020), http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Pythag/pythag.html

This site has a plethora of interesting content on Pythagorean right-angled triangles, including history, tables, explanations, graphs, diagrams, puzzles, problems and some clever and convenient on-line calculators.

[2] I. Pearce, Indian Mathematics - Redressing the balance. Mathematics in the service of religion: II. Sulba Sutras, (accessed 14/3/2020), https://mathshistory.st-andrews.ac.uk/Projects/Pearce/chapter-5/

An intriguing and instructive description of ancient Indian mathematical understanding.

[3] Wikipedia, Pythagorean Theorem, (Accessed 14/3/2020), https://en.wikipedia.org/wiki/Pythagorean_theorem

A useful general introduction and a source of many useful links on history of Pythagoras' Theorem.

[4] R. Gustafson, Was Pythagoras Chinese? - Revisiting an old debate, *The Mathematics Enthusiast* **9** (2012), 207–220.

A fascinating trip through ancient Chinese mathematical know-how.

[5] B. Casselman, The Babylonian tablet Plimpton 322, (Accessed 21/2/2020), https://www.math.ubc.ca/~cass/courses/m446-03/pl322/pl322.html

Recommended if you want to go deeper into the Plimpton 322 inscriptions, the sexagesimals number system and conversion into ordinary decimal numbers.

[6] A.A. Abdulaziz, The Plimpton 322 Tablet and the Babylonian method of generating Pythagorean triples, (Accessed 21/2/2020), https://arxiv.org/pdf/1004.0025.pdf

Another view of the tablet and nice analysis of the numbers and triangles it contains.

[7] D.E. Joyce, Plimpton 322, (Accessed 21/2/2020)
https://mathcs.clarku.edu/~djoyce/ma105/plimpnote.html

Another description of the exact inscriptions on the Tablet, their base-60 meaning, conversion into decimals and analysis of what the columns of figures mean.

[8] E. Robson, Words and Pictures: New Light on Plimpton 322, *The American Mathematical Monthly* **109** (2002), 105–120.

Even deeper into the exact inscriptions on the Tablet.

[9] D.F. Mansfield and N.J. Wildberger, Plimpton 322 is Babylonian exact sexagesimal trigonometry, *Historia Mathematica* **44** (2017), 395–419

More brain-stretching aspects of Babylonian mathematics.

[10] 3Blue1Brown, All possible pythagorean triples, visualized, https://youtu.be/QJYmyhnaaek, 26 May 2017.

There are also heaps of other great resources on Pythagorean triangles available on www.youtube.com. If you're a student, then just reassure your Mum or Dad that you're doing your maths homework! Some of the video content may be too sophisticated (university-level), so you may have to sift through and find things that are explained more simply.