Parabola Volume 56, Issue 2 (2020)

Partial sum of the sequence of aliquot sums

Timothy Hume[1]

1 Introduction

The *aliquot sum* $s(n)$, for a positive integer n, is the sum of all divisors of n, excluding n itself. For example, the aliquot sum of 12 is $1+2+3+4+6 = 16$. The sequence of aliquot sums, $s(n)$ for $n = 1, 2, 3, \ldots$, has been studied by mathematicians since antiquity.

The On-Line Encyclopedia of Integer Sequences (OEIS) [1] presents an elegant formula - without proof - for the sum of the first n terms of the sequence of aliquot sums[2]:

$$\sum_{i=1}^{n} s(n) = \frac{n(n-1)}{2} - \sum_{i=1}^{n} (n \bmod i)\,. \tag{1}$$

In this article a proof of the above formula is presented. However, first a brief introduction to the aliquot sum is presented for readers who may be unfamiliar with this sequence.

2 The aliquot sum

The first few terms in the sequence of aliquot sums are shown in Table 1.

n:	1	2	3	4	5	6	7	8	9	10	11	12
$s(n)$:	0	1	1	3	1	6	1	7	4	8	1	16

Table 1: The first twelve terms in the sequence of aliquot sums.

The sequence can also be visually represented as a scatter plot, as shown in Figure 1. Representing sequences as scatter plots can be helpful, because the eye is adept at finding patterns which might otherwise be missed in long lists of numbers.

A number of patterns are evident in the scatter plot. Not surprisingly, as n increases, $s(n)$ tends to increase, because larger numbers often have more or larger factors. However, small values of $s(n)$ still occur; the extreme case being that all prime numbers have an aliquot sum of 1.

[1]Tim Hume (tim@nomuka.com) works as a meteorologist.

[2]The OEIS presents the partial sum of the sequence of aliquot sums as a difference between two other sequences which are equivalent to Equation 1 above.

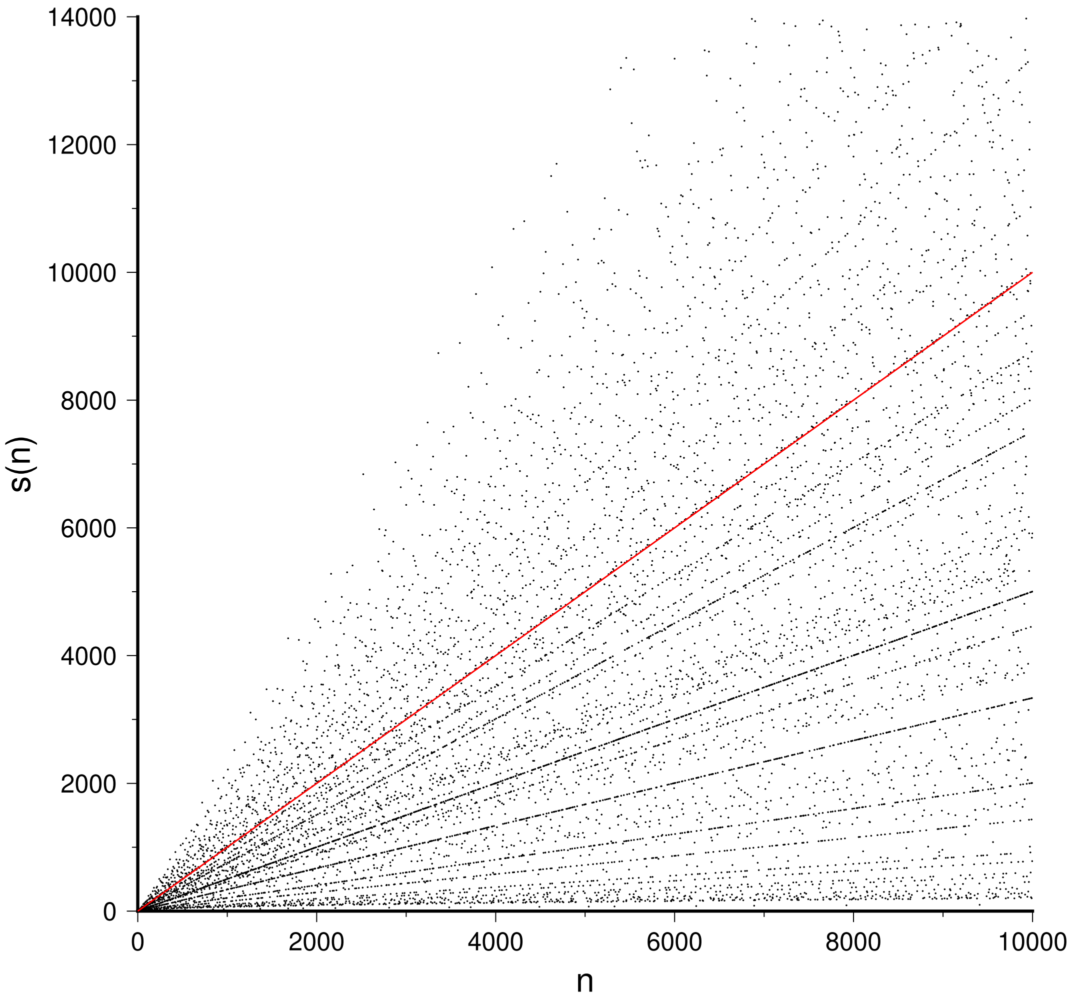

Figure 1: Scatter plot showing the aliquot sum $s(n)$ for $n \leqslant 10000$. Points which fall below the red line are deficient, whilst those above it are abundant. Points which fall exactly on the red line are perfect numbers.

There also appear to be several straight lines of varying angles. To explain at least some of these lines, consider the example of numbers where $n = 2p$, where p are the primes $2, 3, 5, 7, 11, \ldots$. The factors of these numbers must be 1, 2, $^{n}/_{2}$ and n. Therefore, the aliquot sum is: $s(n) = 3 + {}^{n}/_{2}$. A straight line of slope $^{1}/_{2}$ and intercept 3 can be drawn through these points. A similar argument applies to $n = 3p$ and so on.

The aliquot sum can be used to define several classes of numbers, some of which are listed in Table 2.

Class	Definition
Prime numbers	$s(n) = 1$
Perfect numbers	$s(n) = n$
Deficient numbers	$s(n) < n$
Abundant numbers	$s(n) > n$
Quasi-perfect numbers	$s(n) = n + 1$
Almost perfect numbers	$s(n) = n - 1$
Untouchable numbers	n is not the aliquot sum of any other number

Table 2: Definitions for several classes of numbers based on their aliquot sum.

Early mathematicians were particularly interested in *perfect numbers*, defined as numbers where $s(n) = n$. The first few perfect numbers are 6, 28, 496 and 8128; these lie on the red line in Figure 1. Around 300 B.C., Euclid showed showed that if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect [2]. However, it took another two millennia for Euler to show that all even perfect numbers are of the form discovered by Euclid [3]. It is not known if there are any odd perfect numbers, but if they do exist, then they must be greater than 10^{1500} [4].

Numbers where $s(n) < n$ are known as *deficient*, whilst those where $s(n) > n$ are *abundant*. Deficient numbers lie below the red line in Figure 1, whilst abundant numbers lie above it. Deficient numbers appear to be more common than abundant numbers. In the first 10000 terms of the sequence, 7508 terms are deficient.

Quasi-perfect and *almost perfect numbers* are those where the aliquot sum is one greater or one less than n respectively. No quasi-perfect numbers have been found; if they exist they must be greater than 10^{35} [5]. The only known almost perfect numbers are the powers of 2 with non-negative exponents [6].

Untouchable numbers are numbers which are not the aliquot sum of any other number. The first two untouchable numbers are 2 and 5, and were mentioned by the mathematician al-Baghdādī circa 1000 A.D. [7]. It is now known that there are infinitely many untouchable numbers [8].

From the preceding discussion, it is clear that the sequence of aliquot sums contains many interesting features, perhaps explaining its continuing attraction to mathematicians for over two millennia.

3 Partial sum of the sequence of aliquot sums

We now return to the partial sum of the sequence of aliquot sums:

$$\sum_{i=1}^{n} s(n) \, .$$

The scatter plot for this series is shown in Figure 2, for n less than or equal to 1000. The shape resembles a parabola, though when zoomed in (see the inset in Figure 2), small departures from a smooth curve are apparent.

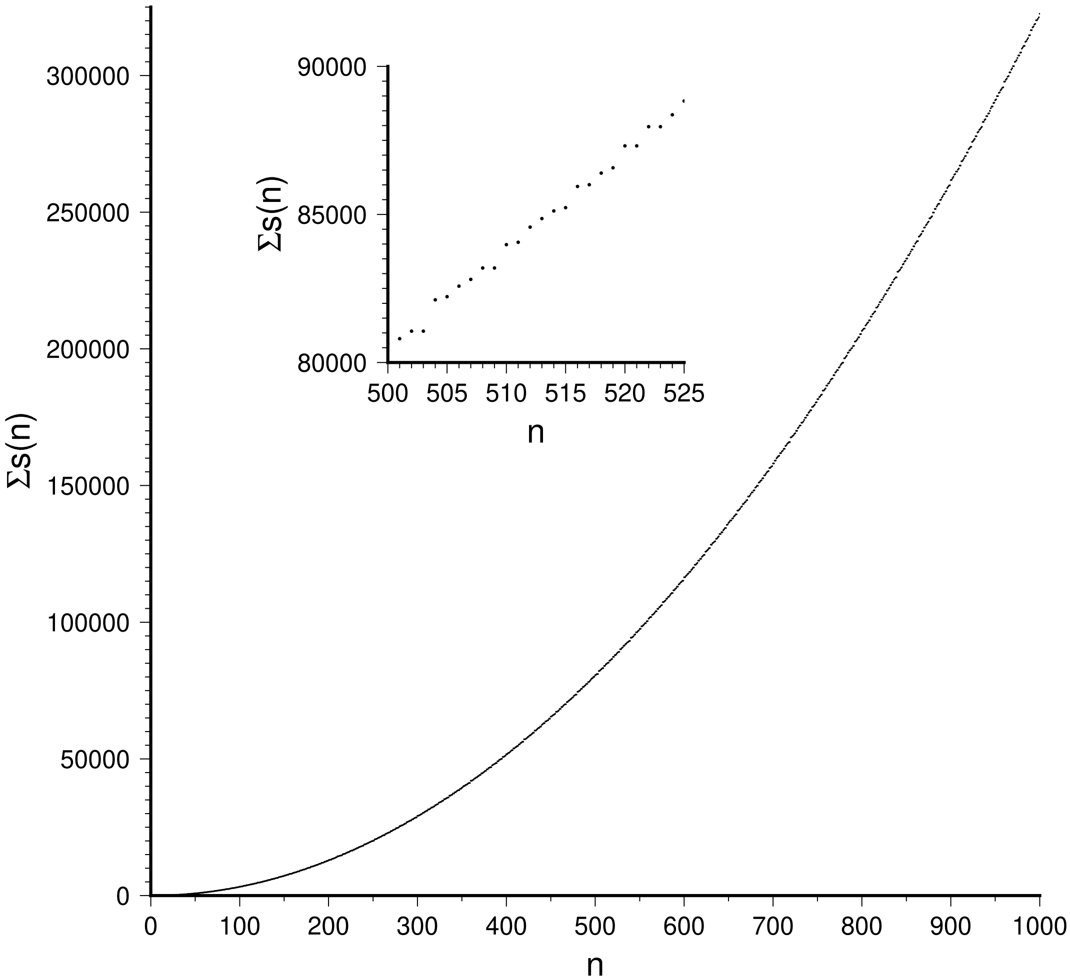

Figure 2: Scatter plot showing the partial sum of the aliquot sum sequence for $n \leqslant 1000$.

To derive Equation 1, which was presented in the introduction, we start by examining the related sequence $m(n)$ defined, for all $n = 1, 2, \ldots$, by

$$m(n) = \sum_{i=1}^{n} (n \bmod i)\,. \tag{2}$$

The $(n-1)^{th}$ term of this sequence is

$$m(n-1) = \sum_{i=1}^{n-1} \big((n-1) \bmod i\big)\,. \tag{3}$$

Now examine the individual terms on the right hand side of Equation 3:

$$(n-1) \bmod i = \begin{cases} (n \bmod i) - 1 & \text{if } n \bmod i > 0\,; \\ i - 1 & \text{if } n \bmod i = 0\,. \end{cases} \tag{4}$$

If n has q factors $f_1, f_2, \ldots, f_q$, where $f_1 = 1$ and $f_q = n$, then

$$n \bmod f_j = 0 \quad \text{for all} \quad 1 \leqslant j \leqslant q\,.$$

By Equation 4,

$$\begin{aligned}\sum_{j=1}^{q-1}\big((n-1) \bmod f_j\big) &= \sum_{j=1}^{q-1}(f_j - 1) \\ &= \left(\sum_{j=1}^{q-1} f_j\right) - (q-1) \\ &= s(n) - q + 1\,. \end{aligned} \tag{5}$$

Having dealt with the factors of n, we now turn our attention to the $n-q$ terms on the right hand side of Equation 3 that are a factor of n, say $g_1, g_2, \ldots, g_{n-q}$:

$$\begin{aligned}\sum_{j=1}^{n-q}\big((n-1) \bmod g_j\big) &= \sum_{j=1}^{n-q}\big((n \bmod g_i) - 1\big) \\ &= \left(\sum_{j=1}^{n-q} n \bmod g_i\right) - n + q \end{aligned} \tag{6}$$

Added, Equations 5 and 6 give the identity

$$\sum_{i=1}^{n-1}\big((n-1) \bmod i\big) = \left(\sum_{j=1}^{n-q} n \bmod g_j\right) + s(n) - n + 1\,. \tag{7}$$

Since $n \bmod f_j = 0$ for $j = 1, 2, \ldots, q$,

$$\begin{aligned}\sum_{i=1}^{n-1}\big((n-1) \bmod i\big) &= \left(\sum_{i=1}^{n-q} n \bmod g_i\right) + \left(\sum_{i=1}^{q} n \bmod f_i\right) + s(n) - n + 1 \\ &= \left(\sum_{i=1}^{n} n \bmod i\right) + s(n) - n + 1\,. \end{aligned}$$

Therefore,

$$m(n-1) = m(n) + s(n) - n + 1\,,$$

or, rearranged,

$$s(n) = m(n-1) - m(n) + n - 1\,. \tag{8}$$

It is now easy to compute the partial sum of $s(n)$:

$$\begin{aligned}\sum_{i=1}^{n} s(n) &= m(0) - m(1) + (1-1)\\ &+ m(1) - m(2) + (2-1)\\ &+ m(2) - m(3) + (3-1)\\ &\vdots\\ &+ m(n-1) - m(n) + (n-1)\,.\end{aligned}$$

Many of the terms in the above equation cancel out, leaving

$$\sum_{i=1}^{n} s(n) = m(0) - m(n) + \sum_{i=1}^{n-1} i\,. \tag{9}$$

Noting that $m(0) = 0$ and that the sum at the far right of Equation 9 is the $(n-1)^{th}$ triangular number given by

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}\,,$$

we have derived Equation 1:

$$\sum_{i=1}^{n} s(n) = \frac{n(n-1)}{2} - \sum_{i=1}^{n} (n \bmod i)$$

4 Final remarks

The proof for the partial sum of the sequence of aliquot sums presented above is but a minor result. However, it was achieved using maths which should be within the grasp of secondary school students. Perhaps this short article may inspire an interest in number theory, a subject which is often not covered in the secondary school curriculum.

After the initial draft of this article was prepared, communication with Thomas Britz inspired an alternative proof of the equation for the partial sum of the sequence of aliquot sums, which is presented in Appendix A. In this proof, another equation for the partial sum is derived:

$$\sum_{i=1}^{n} s(n) = \sum_{i=1}^{n} i \left\lfloor \frac{n}{i} \right\rfloor - \frac{n(n+1)}{2} \tag{10}$$

An interesting question is: What is the most computationally efficient formula for computing the partial sum? Equation 1 requires approximately half the number of operations (multiplication, division, subtraction, addition and so on) to compute the partial sum as Equation 10 (see Table 3). However, it should be noted that some operations, such as division, are more computationally expensive than operations such as addition. The exact amount of computational effort for an operation such as division depends on the algorithm used, and the design of the computer hardware. The author is aware of at least one modification to Equation 1 which reduces the number of computations required to compute the partial sum by almost half again. Are there even more computationally efficient ways of computing the partial sum?

	Number of times operation is used	
Operation	Equation 1	Equation 10
Addition	n	$n+1$
Subtraction	2	1
Multiplication	1	$n+1$
Division	1	$n+1$
Modulo	n	0
Floor	0	n
Total	$2n+4$	$4n+4$

Table 3: Number of arithmetic operations required to compute the partial sum of the sequence of aliquot sums using Equations 1 and 10.

There are straight lines visible in Figure 1 which cannot be understood using the simple explanation presented in Section 2 (for example, the lines which can be seen running through the abundant numbers above the red line in the figure). It may be interesting to look for a more general explanation for these lines. A related question may be whether there are any "lone" points in the scatter plot which are not part of any line?

On a final note, Equation 3 reduces to

$$m(n-1) = m(n) + 1 \tag{11}$$

when n is a perfect number, because $s(n) = n$. The author spent too much time contemplating whether Equation 11 can be solved when n is an odd number. If it can (or cannot) be solved, this would be a proof of the existence (or not) of odd perfect numbers.

Acknowledgements

I would like to thank Dr Thomas Britz for encouraging me to write this article, and for his helpful comments on the draft. Figures 1 and 2 were prepared using the open source Generic Mapping Tools [9].

A Alternative proof of the equation for the partial sum of the sequence of aliquot sums

Consider all the divisors of all the numbers $1, 2, \ldots, n$. These divisors must also be between 1 and n inclusive. The numbers which have each divisor between 1 and n are listed in Table 4.

i	Numbers $\leqslant n$ where i is a divisor				
1	$1 \times 1,$	$2 \times 1,$	$3 \times 1,$	$\ldots,$	$\left\lfloor \frac{n}{1} \right\rfloor \times 1$
2		$1 \times 2,$	$2 \times 2,$	$\ldots,$	$\left\lfloor \frac{n}{2} \right\rfloor \times 2$
$\vdots$					
i			$1 \times i,$	$\ldots,$	$\left\lfloor \frac{n}{i} \right\rfloor \times i$
$\vdots$					
n					$\left\lfloor \frac{n}{n} \right\rfloor \times n$

Table 4: Divisors of all the numbers less than or equal to n.

The total number of divisors for all the numbers between 1 and n inclusive is the number of non-blank entries in Table 4. In the first row there are $\left\lfloor \frac{n}{1} \right\rfloor$ entries, in the second row there are $\left\lfloor \frac{n}{2} \right\rfloor$ entries, and so on until there is $\left\lfloor \frac{n}{n} \right\rfloor$, or 1, entry in the last row. That is,

$$\sum_{i=1}^{n} \sigma_0(i) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

where $\sigma_0(i)$ is the number of divisors of i.

The sum of all the divisors of the numbers between 1 and n inclusive can be computed in a similar way. In the first row of the table, all the divisors are 1, so the sum of the first row is $1 \times \left\lfloor \frac{n}{1} \right\rfloor$. For the second row the sum is $2 \times \left\lfloor \frac{n}{2} \right\rfloor$ and so on. The total sum is

$$\sum_{i=1}^{n} \sigma_1(i) = \sum_{i=1}^{n} i \left\lfloor \frac{n}{i} \right\rfloor \tag{12}$$

where $\sigma_1(i)$ is the sum of all the divisors of i, including i.

To get the sum of the aliquot sums between 1 and n inclusive we have to subtract the numbers in Table 4 which are equal to the divisors. These are the entries on the diagonal of the table:

$$1 \times 1\,,\ 1 \times 2\,,\ \ldots\,,\ \left\lfloor \frac{n}{n} \right\rfloor \times n\,.$$

Subtracting these numbers form Equation 12 gives

$$\sum_{i=1}^{n} s(n) = \sum_{i=1}^{n} \sigma_1(i) - \sum_{i=1}^{n} i$$
$$= \sum_{i=1}^{n} i \left\lfloor \frac{n}{i} \right\rfloor - \frac{n(n+1)}{2} \tag{13}$$

We now note that

$$i \left\lfloor \frac{n}{i} \right\rfloor = n - n \bmod i \,. \tag{14}$$

Substituting Equation 14 into Equation 13 and rearranging gives Equation 1 once again:

$$\sum_{i=1}^{n} s(n) = \frac{n(n-1)}{2} - \sum_{i=1}^{n} (n \bmod i) \,.$$

References

[1] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org/A153485, 2020.

[2] Euclid, *The Thirteen Books of Euclid's Elements*, translated by T.L. Heath, Cambridge University Press, vol. 2, 1908.

[3] L. Euler, De numeris amicabilibus, in *Opera Postuma mathematica et physica*, p. 88, Saint Petersburg Academy of Science, 1862. http://eulerarchive.maa.org//docs/originals/E798.pdf

[4] P. Ochem and M. Rao, Odd perfect numbers are greater than 10^{1500}, *Math. Comp.* **81** (2012), 1869–1877.

[5] P. Hagis, Jr. and G.L. Cohen, Some results concerning quasiperfect numbers, *J. Austral. Math. Soc. Ser. A* **33** (1982), 275–286.

[6] R.K. Guy, *Unsolved problems in Number Theory*, 2nd edition, Springer-Verlag, New York, 1994.

[7] J. Sesiano, Two problems of number theory in Islamic times, *Arch. Hist. Exact Sci.* **41** (1991), 235–238.

[8] P. Erdős, Über die Zahlen der Form $\sigma(n) - n$ und $n - \phi(n)$, *Elemente der Mathematik* **28** (1973), 83–86.

[9] P. Wessel, J.F. Luis, L. Uieda, R. Scharroo, F. Wobbe, W.H.F. Smith and D. Tian, The Generic Mapping Tools version 6, *Geochemistry, Geophysics, Geosystems* **20** (2019), 5556–5564.