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Problems 1621–1630

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Q1621 Take a point O inside a square; from this point draw six rays, all spaced at equal angles. This will divide the square into six regions. Is it possible that all these regions have equal area?

Q1622 Find the sum of the digits of

$$S = 1 + 11 + 111 + 1111 + \cdots + \overbrace{11\cdots11}^{999\text{ digits}},$$

where the last term on the right hand side has 999 digits, all equal to 1.

Q1623 Consider the product of n factors

$$P_n = 4^{1/4}\,16^{1/16}\,64^{1/64}\cdots(4^n)^{1/4^n}\,.$$

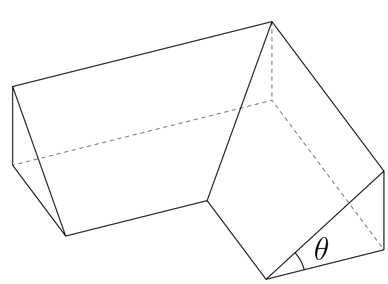
Show that if the number of factors n becomes larger and larger, the product gets closer and closer to a fixed value; and find this value.

Q1624 A semicircle has diameter AB. Two chords AC and BD meet at a point X outside the semicircle. Prove that

$$(AC)(AX) + (BD)(BX) = (AB)^2\,.$$

Q1625 Find the smallest multiple of 4321 which ends in the digits 1234. *Hint*: just find a suitable multiple first. To investigate whether or not it is the smallest possible, see the article *Linear Diophantine Equations* in *Parabola* Volume 49 Issue 2, which can be found here.

Q1626 I want to join at right angles two iron roofs pitched at an angle θ (see the diagram: the upper and lower edges of each roof are to be parallel). At what angles do I need to cut the iron pieces?



Q1627 Find the number of solutions to

$$3x + 2y + z = 2020$$

where x, y and z are positive integers.

Q1628 A path is marked with the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. A person stands at the point 0 and flips a coin (a fair coin, so that heads and tails come up with probability $\frac{1}{2}$). If the flip is heads, then the person moves one step in the positive direction; if tails, then he moves one step in the negative direction. This is done repeatedly. Denote by p_n the probability that the person is back at the position 0 after n steps.

(a) Explain why $p_n = 0$ if n is odd.

(b) Compute the probability p_{2k} of returning to position 0 after $n = 2k$ steps.

Q1629 We have a bag and seven slips of paper on which are written the following statements:

- at least one of the statements in the bag is true;
- at least two of the statements in the bag are true;
- at least three of the statements in the bag are true;
- at least one of the statements in the bag is false;
- at least two of the statements in the bag are false;
- at least three of the statements in the bag are false;
- at least four of the statements in the bag are false.

One of the slips is removed; the other six are placed in the bag and we determine whether they are true or false. How many are false?

Q1630 Recall from Problem 1617 that F is the set of all *dyadic fractions*, that is, fractions in which the denominator is a power of 2; and that for any set X, we write $aX + b$ for the set of all numbers which can be written $ax + b$, where x is in X.

We say that a set is *locally finite* if it has only finitely many elements in any finite interval of the real number line. For example, the set of all integers is not finite, but it is *locally* finite since any interval $p < x < q$ contains only finitely many integers.

(a) Prove that F is not locally finite.

If A and B are sets of numbers, the *symmetric difference* $A \oplus B$ is the set of all numbers which are in A or B but not both. For example,

$$\{1, 2, 3, 4, 5\} \oplus \{4, 5, 6, 7, 8\} = \{1, 2, 3, 6, 7, 8\}.$$

The numbers $4, 5$ are not included in the right hand side because they are in *both* sets on the left hand side.

We consider subsets X of F with the property

$$(2X) \oplus X \text{ and } (X + \tfrac{1}{2}) \oplus X \text{ are both locally finite.}$$

Since such X are important in a current mathematical research project, we shall call them "important" sets.

(b) Prove that if X is a subset of F (that is, X is some set of dyadic fractions) and X is locally finite, then X is important.

(c) Let X be a subset of F and suppose that $\overline{X}$, the set of all dyadic fractions **not** in X, is locally finite. Show that X is important.