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Strange irrational powers

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The number $1+\sqrt{2}$ has an interesting property. When we calculate the sequence

$$(1+\sqrt{2})^1, (1+\sqrt{2})^2, (1+\sqrt{2})^3, \ldots$$

the results seem to get closer and closer to an integer (i.e., the numbers $\ldots, -2, -1, 0, 1, 2, \ldots$). Let's work out the first 10 powers and see:

Expression	Calculation	Distance to nearest integer
$(1+\sqrt{2})^1$	2.41421	0.41421
$(1+\sqrt{2})^2$	5.82843	0.17157
$(1+\sqrt{2})^3$	14.07107	0.07107
$(1+\sqrt{2})^4$	33.97056	0.02944
$(1+\sqrt{2})^5$	82.01219	0.01219
$(1+\sqrt{2})^6$	197.99495	0.00505
$(1+\sqrt{2})^7$	478.00209	0.00209
$(1+\sqrt{2})^8$	1153.99913	0.00087
$(1+\sqrt{2})^9$	2786.00036	0.00036
$(1+\sqrt{2})^{10}$	6725.99984	0.00016

Table 1: Distance of $(1+\sqrt{2})^n$ to nearest integer

Is it true that $(1+\sqrt{2})^n$ gets closer to an integer as n increases?

We'll come back to that question soon. Let's see if we can find another number with this property. We are only interested in the distance to the nearest integer, so going forward we can delete the column with the calculation of the irrational raised to the powers. We will just show the distance to the nearest integer.

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This time we'll try $1+\sqrt{3}$ to increasing powers:

Expression	Distance to nearest integer
$(1+\sqrt{3})^1$	0.26795
$(1+\sqrt{3})^2$	0.46410
$(1+\sqrt{3})^3$	0.39230
$(1+\sqrt{3})^4$	0.28719
$(1+\sqrt{3})^5$	0.21024
$(1+\sqrt{3})^6$	0.15390
$(1+\sqrt{3})^7$	0.11267
$(1+\sqrt{3})^8$	0.08247
$(1+\sqrt{3})^9$	0.06039
$(1+\sqrt{3})^{10}$	0.04416

Table 2: Distance of $(1+\sqrt{3})^n$ to nearest integer

The distance goes up from $(1+\sqrt{3})^1$ to $(1+\sqrt{3})^2$ but then establishes a downward trend. But it doesn't work for all of these type of numbers. Let's try with $(1+\sqrt{7})$:

Expression	Distance to nearest integer
$(1+\sqrt{7})^1$	0.35425
$(1+\sqrt{7})^2$	0.29150
$(1+\sqrt{7})^3$	0.45751
$(1+\sqrt{7})^4$	0.33596
$(1+\sqrt{7})^5$	0.07316
$(1+\sqrt{7})^6$	0.13058
$(1+\sqrt{7})^7$	0.29987
$(1+\sqrt{7})^8$	0.18372
$(1+\sqrt{7})^9$	0.43180
$(1+\sqrt{7})^{10}$	0.23870

Table 3: Distance of $(1+\sqrt{7})^n$ to nearest integer

The expression $(1+\sqrt{7})^n$ doesn't get closer to an integer as n increases. And there is nothing special about having the 1 that we used in $1+\sqrt{2}$, $1+\sqrt{3}$ and $1+\sqrt{7}$. For example, $4+\sqrt{18}$ works really well:

Expression	Distance to nearest integer
$(4+\sqrt{18})^1$	0.2426407
$(4+\sqrt{18})^2$	0.0588745
$(4+\sqrt{18})^3$	0.0142853
$(4+\sqrt{18})^4$	0.0034662
$(4+\sqrt{18})^5$	0.0008410
$(4+\sqrt{18})^6$	0.0002041
$(4+\sqrt{18})^7$	0.0000495
$(4+\sqrt{18})^8$	0.0000120
$(4+\sqrt{18})^9$	0.0000029
$(4+\sqrt{18})^{10}$	0.0000007

Table 4: Distance of $(4+\sqrt{18})^n$ to nearest integer

Let's go back to the central question of this article.

Does $1+\sqrt{2}$ get closer and closer to an integer as we raise it to increasing powers?

Let's try expanding $(1+\sqrt{2})^2$. We have

$$(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + (\sqrt{2})^2 = 1 + 2\sqrt{2} + 2\,.$$

The only non-integer term on the right hand side is $2\sqrt{2}$. If we could reduce the term $\sqrt{2}$, then we would definitely get closer to an integer. This is where mathematics gets creative. Let's try adding $(1-\sqrt{2})^2$ to both sides. This gives us

$$(1+\sqrt{2})^2 + (1-\sqrt{2})^2 = (1+2\sqrt{2}+2) + (1-2\sqrt{2}+2) = 2+4\,.$$

We have removed the square root! Let's try using powers of three.

$$(1+\sqrt{2})^3 + (1-\sqrt{2})^3 = (1+3\sqrt{2}+3\cdot 2+2\sqrt{2}) + (1-3\sqrt{2}+3\cdot 2-2\sqrt{2}) = 14\,.$$

In fact, this works for any power. Using the Binomial Theorem, we have

$$\begin{aligned}(1+\sqrt{2})^n + (1-\sqrt{2})^n &= 1 + \binom{n}{1}\sqrt{2} + \binom{n}{2}(\sqrt{2})^2 + \cdots + \binom{n}{n}(\sqrt{2})^n \\ &\quad + 1 - \binom{n}{1}\sqrt{2} + \binom{n}{2}(\sqrt{2})^2 + \cdots + (-1)^n\binom{n}{n}(\sqrt{2})^n \\ &= 2 + 2\binom{n}{2}2 + 2\binom{n}{4}4 + \cdots + \binom{n}{n}(1+(-1)^n)(\sqrt{2})^n\,.\end{aligned}$$

That last term will equal zero if n is odd and $2\cdot 2^{\frac{n}{2}}$ if n is even; either way, it is an integer.

Have a look at the right-hand side again,

$$2+2\binom{n}{2}2+2\binom{n}{4}4+\cdots+\binom{n}{n}(1+(-1)^n)(\sqrt{2})^n\,.$$

There's not a square root in sight! It's even better than that. All the terms of the form $\binom{n}{r}$ are just the number of ways of choosing r objects from n objects. Since this is a counting process, $\binom{n}{r}$ is an integer. So this means that every term on the right-hand side of the equation above is an integer. The right-hand side is just the result of adding and multiplying integers. So it must be an integer. If the right-hand side is an integer, then the left hand side must an integer. I'll say that again because it is important.

The expression $(1+\sqrt{2})^n+(1-\sqrt{2})^n$ *is an integer.*

We are nearly there. We just need one more thing. The number $1-\sqrt{2}$ is equal to -0.4142, approximately. Whenever we raise a number between -1 and 1 to increasing powers, the number gets closer and closer to zero. Let's calculate $(1-\sqrt{2})^n$ and we will see.

Expression	Distance to nearest integer
$(1-\sqrt{2})^1$	-0.41421
$(1-\sqrt{2})^2$	0.17157
$(1-\sqrt{2})^3$	-0.07107
$(1-\sqrt{2})^4$	0.02944
$(1-\sqrt{2})^5$	-0.01219
$(1-\sqrt{2})^6$	0.00505
$(1-\sqrt{2})^7$	-0.00209
$(1-\sqrt{2})^8$	0.00087
$(1-\sqrt{2})^9$	-0.00036
$(1-\sqrt{2})^{10}$	0.00015

Table 5: Distance of $(1-\sqrt{2})^n$ to zero

Let's summarise where we have got to. We know that

$$(1+\sqrt{2})^n+(1-\sqrt{2})^n$$

is an integer. And we know that $(1-\sqrt{2})^n$ gets closer and closer to zero as n grows larger. So this means that $(1+\sqrt{2})^n$ must get closer and closer to an integer.

Another way to look at this is to create the following equation with $(1+\sqrt{2})^n$ on the left hand side:

$$(1+\sqrt{2})^n=\underbrace{(1+\sqrt{2})^n+(1-\sqrt{2})^n}_{\text{an integer}}-\underbrace{(1-\sqrt{2})^n}_{\text{gets closer to zero}}\quad.$$

This shows that $(1+\sqrt{2})^n$ gets closer and closer to an integer.

I hope you can now see why $(1+\sqrt{3})^n$ gets closer and closer to an integer. It's because $1-\sqrt{3}$ is between -1 and 1. It also explains why $(1+\sqrt{5})^n$ doesn't get closer and closer to an integer. It's because $1-\sqrt{5}$ is approximately equal to -1.2361 and -1.2361 is not between -1 and 1.

I'll make one more comment before we finish. Many Parabola readers will be familiar with Fibonacci numbers (there were many articles about Fibonacci numbers in an earlier edition[2] of Parabola this year). The Fibonacci numbers are the numbers

$$0, 1, 1, 2, 3, 5, 8, \ldots .$$

where the first term is 0, the second term is 1 and the remaining terms are the sum of the previous two terms. The formula for the n-th term in the sequence of Fibonacci numbers is given by the following:

$$F_n = \frac{1}{5}\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \frac{1}{5}\Big(\frac{1-\sqrt{5}}{2}\Big)^n .$$

Every Fibonacci number after the first two is just the sum of two integers. So the Fibonacci numbers are all integers. So the left hand side is an integer. Also the term

$$\frac{1}{5}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

gets closer and closer to zero as n increases. This means that F_n gets closer and closer to

$$\frac{1}{5}\Big(\frac{1+\sqrt{5}}{2}\Big)^n .$$

So if you ever need to work out the Fibonacci numbers, then ignore the term

$$\frac{1}{5}\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

in the formula and just work out the closest integer to

$$\frac{1}{5}\Big(\frac{1+\sqrt{5}}{2}\Big)^n .$$

For example, what is F_{25}? Well,

$$\frac{1}{5}\Big(\frac{1+\sqrt{5}}{2}\Big)^{100} = 75024.9999973342\ldots$$

So the 25th Fibonacci number is 75,025.

[2] https://www.parabola.unsw.edu.au/2020-2029/volume-56-2020/issue-1