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Problems 1631–1640

Parabola would like to thank Toyesh Prakash Sharma for contributing Problem 1635, and Shiva Oswal for a problem which inspired Problem 1636.

Q1631 As in Problem 1628, a path is labelled with the integers, and a person starting at 0 moves one step at a time left or right according to the flip of a fair coin. In the previous problem (whose solution appears in this issue), we found the probability that the person is back at the origin after $2n$ steps. Now see if you can find the probability that the person is back at the origin **for the first time** after $2n$ steps.

Q1632 Take a point O inside a square; from this point draw 13 rays, all spaced at equal angles. This will divide the square into 13 regions. Is it possible that all these regions have equal area?

Q1633 A right circular cone (with a closed base) is partially filled with water. The base of the cone is placed on a table and the depth of water in the cone is found to be 10cm. The cone is then inverted so that its vertex is on the table and its base is parallel to the table, and the depth of water is found to be 11cm. What height of empty space is there now above the water surface?

Q1634 Let n be a positive integer which is the sum of the squares of $2k$ positive integers, and suppose that no more than half of these squares are the same. Prove that $2n$ is also the sum of the squares of $2k$ positive integers. For example,

$$65 = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 5^2$$
$$130 = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 8^2\ .$$

Conversely, let $2m$ be an even number which is the sum of the squares of $2k$ positive integers. Suppose that no more than half of the odd squares are the same and no more than half of the even squares are the same. Prove that m is also the sum of the squares of $2k$ positive integers.

Q1635 Suppose that

$$x = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \cdots + \frac{1}{2021\times 2022}$$

and

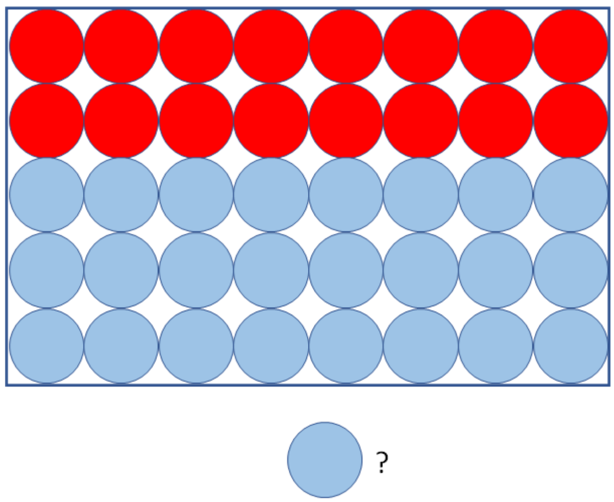
$$y = \frac{1}{1012\times 2022} + \frac{1}{1013\times 2021} + \frac{1}{1014\times 2020} + \cdots + \frac{1}{2022\times 1012}\ .$$

Find the value of x/y.

Q1636 In a right–angled triangle, a string with length equal to the hypotenuse is laid along the longer side, with the excess then going around the right angle and being laid along part of the shorter side. Find the maximum possible proportion of the shorter side covered by string, and the triangles which yield this maximum.

Q1637 Find the smallest multiple of 8642 which ends in the digits 2468.

Q1638 A toy game designer devised a game consisting of 41 solid marbles. She noticed that a rectangular box for 40 marbles, packed as shown, is only $\pi/4 \approx 79\%$ full. How did she manage to fit one more marble into the box?



Q1639 What is the largest integer that **cannot** be expressed as $99a+100b+101c$, where a, b, c are non–negative integers?

Q1640 Recall that in Problems 1617 and 1630 we made the following definitions:

- F is the set of all dyadic fractions, that is, fractions in which the denominator is a power of 2; and for any set X, we write $aX + b$ for the set of all numbers which can be written $ax + b$, where x is in X.
- A set is *locally finite* if it has only finitely many elements in any finite interval of the real number line.
- A set X is called "important" if

$$(2X) \oplus X \text{ and } (X + \tfrac{1}{2}) \oplus X \text{ are both locally finite,}$$

  where $(2X) \oplus X$ denotes the set of numbers which are in $2X$ or X but not both, and similarly for the second expression.

Now if a is any integer and m is a positive integer, we write $S_{a,m}$ for the set of all dyadic fractions $s/2^n$ such that s has remainder a when divided by m: that is,

$$S_{a,m} = \left\{ \frac{s}{2^n} \;\middle|\; n \geq 0 \text{ and } s \text{ divided by } m \text{ has remainder } a \right\}.$$

(a) Show that none of the sets $S_{a,m}$ is locally finite.

(b) Show that none of the sets $\overline{S_{a,m}}$ is locally finite.

(c) Show that if a is odd and m is a power of 2 (that is, $m = 2^l$ for some $l \geq 1$), then $S_{a,m}$ is important; and that no two of the sets

$$S_{3,4}\ ,\ S_{5,8}\ ,\ S_{9,16}\ \ldots,\ S_{2^k+1,2^{k+1}}$$

have any elements in common.

Comment. The point of the question is this: we proved last issue that if either X or $\overline{X}$ is locally finite, then X is important. The present question shows that there are further types of important sets which we have not yet seen.