Parabola Volume 57, Issue 1 (2021)

Vertical oscillations of a bridge induced by a pedestrian

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Abstract

Pedestrians have been known to cause vibrations on bridges from the forces exerted by their footsteps as they cross it. In the case of soldiers, squadrons are ordered to break stride to prevent marching in unison while crossing the bridge. This is to avoid causing more significant vibrations that have the potential to collapse the bridge. This is an example of a pervasive phenomenon known as resonance. We review, develop, and analyze a mathematical model of the coupled dynamics of the bridge and footsteps of soldiers that illustrates the phenomenon of resonance.

1 Introduction

In several instances of scientific literature, we can find cases of bridges collapsing as a result of soldiers marching across it [1, 2, 3]. The fundamental principles that cause the collapse of the bridge are already known. In order to better understand the physical representation of this phenomenon, we consider the bridge as a simple, harmonic oscillator. The bridge is assumed to remain restricted to vertical motion, following Hooke's Law [4]. This is illustrated in Figure 2, where we model the bridge as a platform atop springs to simulate the behavior of the bridge's oscillations.

To understand the motion and characteristics of a simple harmonic oscillator, we consider a more familiar system. For this example, we refer to a toddler on a swing. All physical objects exist with a natural frequency. The pendulum that is the swing has a natural frequency. In a controlled environment with no external factors affecting the state of the pendulum, assuming the pendulum has an initial motion, it will oscillate at a specific frequency that is dependent on the properties of the pendulum itself. With this particular environment and conditions, the swing will move back and forth in a repeated pattern without disruption. In a real world application, the amplitude of the oscillations will gradually decrease as time passes and the pendulum will eventually stop oscillating. The decrease in amplitude is a result of damping, which is a combination of various factors like friction and gravity that exert forces that go against the forces of the oscillations. The bridge behaves similarly, though the oscillations are in the vertical direction and the force required to set the bridge in motion is slightly larger than the force to move the toddler.

Once the toddler is pushed on the swing, it becomes an example of a forced oscillator. The exerted force is external, by the person pushing the toddler. The force is

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periodic, and the frequency of the forces are synchronous with the oscillations of the swinging toddler. The child is pushed at the same point during each oscillate, and the toddler will continue to swing higher and higher. This is an example of a phenomenon known as resonance. Though the oscillations will continue to increase indefinitely, in a real-world scenario, damping would limit the oscillations.

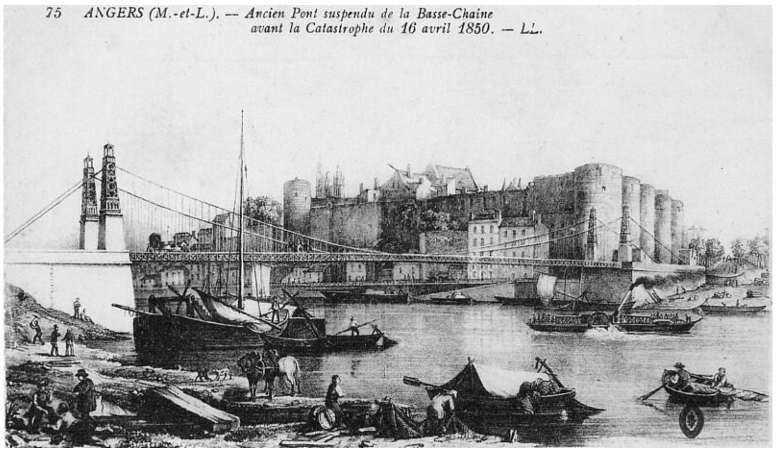

Figure 1: Image of Pont de la Basse-Chaîne (Angers bridge) on 23 November 2007, prior to the collapse in 1850.

The physics of the oscillations of a bridge caused by the marching of soldiers is comparable to the physics of the oscillations of a toddler on a swing being pushed by a person. Similar to the intervals of the person's pushes of the toddler, the periodic force of the footsteps of the soldiers will cause the bridge to oscillate. When the forces of the footsteps are in resonance, the amplitude of the oscillations will continue to increase over time. Though damping will hinder the amplitude of the oscillations, it may not reduce the impacts of the soldiers' footsteps enough to avoid the collapsing of the bridge.

This phenomenon of resonance is the subject of study for this article. The article is organized as such: the model of the bridge is elaborated in Section II, the mathematical model of the instance is recognized in Section III, the natural motion of the bridge is identified in Section IV, and the impacts of the soldiers on the bridge is established in Section V.

2 Mathematical Model of a Bridge Without Pedestrians

As previously mentioned in the Introduction, the bridge is to be considered a platform restricted to vertical motion only. The model also represents the vibrations or shaking of the bridge with the platform being on top of a spring. In the model, we disregard the effects of damping. The only acting forces on the platform are the weight of the platform and the elastic force of the platform and spring, denoted by F_e. The mass of the platform is denoted by M. This representation is modeled following Hooke's

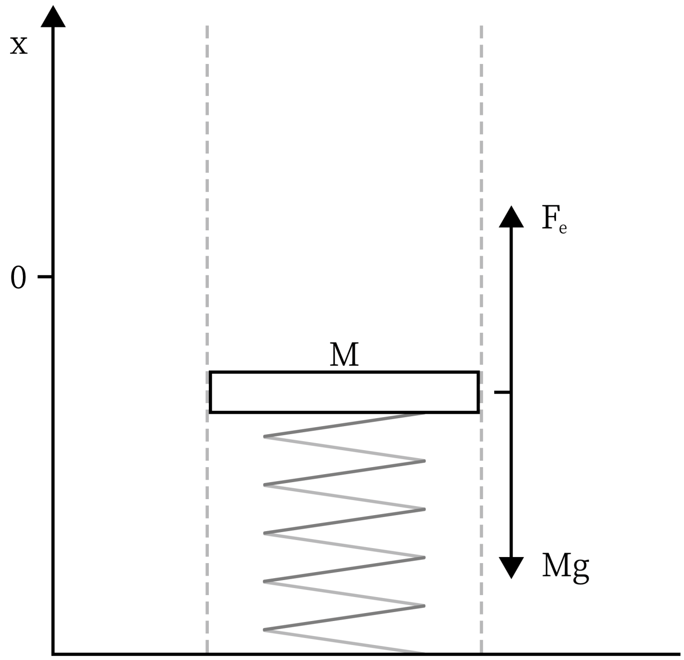

Figure 2: Bridge as a platform. Free body diagram.

Law. At a specific height value, dependent on the weight and elastic property of the platform, the force is zero. This height is called the equilibrium height. If the height is at any other point, the elastic force is inversely proportional to the difference of this height and the equilibrium height. For instance, if the platform is below the equilibrium height, the elastic force will be a positive force, that goes in the upward direction.

We denote by x the height of the platform along a vertical axis, when $x = 0$. The positive x-values are the positions at which the platform is above the equilibrium height and the negative values are below. The elastic force of the platform can be defined as $F_e = -kx$, where k is a positive constant that represents a elastic property of the spring. The negative sign creates a resisting force that acts in the opposite direction that the platform is in relative to the equilibrium point. The x-value creates a larger elastic force as it moves further away from the equilibrium point. By denoting g the gravitational constant and a the acceleration of the platform, we can use Newton's second law of motion, force equals mass times acceleration, to create the equation

$$-kx - Mg = Ma\,. \tag{1}$$

We consider the change in the bridge's x-position over time. We use the standard notation of t to denote time and v to denote the velocity of the platform. The functions x, v, and a are all functions of t. We recall that the function a is the derivative of function v, and that the function v is the derivative of function x. We denote the derivatives of functions with primes. As a result, we can replace the a in Equation (1), which then becomes

$$-kx - Mg = Mx''\,. \tag{2}$$

Equation (2) is a different equation where the unknown is a function. In the case of this equation, it would be $x = x(t)$. Derivatives of the unknown also appear in the equation, which creates an infinite number of solutions to the equation. There must be defined variables that allow for only one possible solution. The required variables are the initial height and velocity of the platform. This can be mathematically described as the values of $x(0)$ and $x'(0)$. It must also be considered that the constant function $x(t) = -Mg/k$ is a solution. This equation represents when the platform is not moving and is at the particular height where the elastic force is of the same magnitude as the weight, though the weight is in the opposite direction, which counteracts the magnitude of each force. With this particular height in mind, there can be a substitution in variables to make future equations simpler. There can be a change in the dependent variable

$$z(t) = x(t) + \frac{Mg}{k}\,. \tag{3}$$

This creates a setting for Equation (2) that sets the height to an equilibrium between the weight of the platform and the elastic magnitude. After algebraic manipulation, and condensing $\sqrt{k/M}$ into the variable w, Equation (2) becomes

$$z'' + w^2 z = 0\,. \tag{4}$$

After arriving at Equation (4), there are an infinite number of solutions to the equation. Though the arrival at the the solution is not necessary, the solutions are

$$z = A\sin(wt + \phi)\,. \tag{5}$$

In Equation (5), the constants are A and ϕ. The constant A is called the amplitude. The solutions to Equation (4) is dependent on the value assigned to those constants. For each combination of constants, there is a solution. Figure 3 shows the plot of the solution when $A = 1, w = 2\pi, \phi = -\pi/4$.

Note that Figure (3) displays the dynamics of the platform based on the assumptions made in the introduction. We neglect the effects of damping, which allows the platform to oscillate with a constant amplitude A and frequency w indefinitely. The frequency is dependent on the natural properties of the platform and the spring that it is on, more precisely, $w = \sqrt{k/M}$. The amplitude A is not dependent on the properties of the system, but on the conditions of the situation.

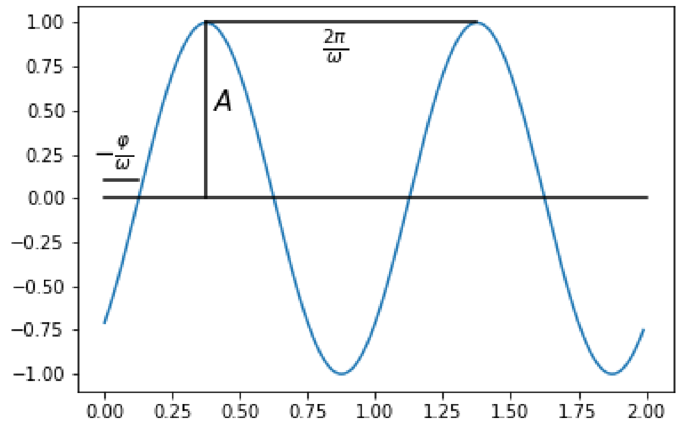

Figure 3: Plot of $z = A\sin(wt + \phi)$ with $A = 1, w = 2\pi, \phi = -\pi/4$.

3 Effect of an Impulse on the Dynamics of the Platform

When we describe an impulse, we refer to a constant force F which is exerted on an object for a brief period of time. We can define the impulse on this object as

$$J = F\Delta t\,. \tag{6}$$

We also note that the momentum of an object, denoted by p, is defined as

$$p = mv\,, \tag{7}$$

where v is the velocity of the object and m is the mass of the object. By Newton's Second Law of Motion, we know that when an object is subjected to an impulse, the change in the momentum of the object is equal to the impulse. If the momentum of the object immediately preceding and following the impulse are p_1 and p_2, respectively, then

$$p_2 - p_1 = J\,. \tag{8}$$

We now bring our attention back to the mechanics of the bridge. As we recall from the previous section, the position of the platform as a function of time t is $z = A\sin(wt + \phi)$, granted, the platform is not subjected to any external forces of any kind. The constants A and ϕ dictate the one solution to Equation (5). As the platform is introduced to an external impulse, Equation (5) becomes no longer valid. Assume the platform is subjected to an impulse $-J$ at a set time when $t = t_1$. The formula $z = A\sin(wt + \phi)$ becomes valid only at time intervals when $t! = t_1$. There will be different intervals that are separated by each impulse that occurs, and each time interval between the impulses will have its own set constants A and ϕ, based on the timing and magnitude of the impulse. This is because the impulse at $t = t_1$ will change the momentum, and thus velocity, of the platform. Similar to how we divided the time intervals of t, we denote $A = A_1$ and $\phi = \phi_1$ as the values of these constants before the impulse and $A = A_2$ and $\phi = \phi_2$ as the values after the impulse. With these new values, we have

$$z = \begin{cases} A_1 \sin(wt + \phi_1) & \text{if } t < t_1 \\ A_2 \sin(wt + \phi_2) & \text{if } t > t_1 \,. \end{cases} \tag{9}$$

We can obtain the constants A_2 and ϕ_2 in terms of A_1 and ϕ_1. We can do this because we know that the position z remains continuous in the moment of the impulse, since the amount of time elapsed between A_1 and A_2 or ϕ_1 and ϕ_2 is 0. We also know that the change in momentum is equal to $-J$ at $t = t_1$. With these facts, we can create the two following equations respectively

$$A_2 \sin(wt_1 + \phi_2) = A_1 \sin(wt_1 + \phi_1) \tag{10}$$

$$MwA_2 \cos(wt_1 + \phi_2) = MwA_1 \cos(wt_1 + \phi_1) - J \,. \tag{11}$$

For the purposes of simplifying the equation that we are about to produce, we define the constant

$$I = J/Mw \,. \tag{12}$$

After dividing the second of the Equations (10) by Mw, the system of equations in Equation (10) becomes

$$A_2 \sin(wt_1 + \phi_2) = A_1 \sin(wt_1 + \phi_1) \tag{13}$$

$$A_2 \cos(wt_1 + \phi_2) = A_1 \cos(wt_1 + \phi_1) - I \,. \tag{14}$$

This new system of equations allows us to define A_2 in terms of A_1 and ϕ_1. When we square both equations, add them together, and take the square root, we are left with

$$A_2 = \sqrt{A_1^2 - 2A_1 I \cos(wt_1 + \phi_1) + I^2} \,. \tag{15}$$

Now that we have created this equation for A_2, we must find ϕ_2. In order to do that, we refer back to Equation (13). When $A_2 = 0$, we can choose any value for ϕ_2. If $A_2 \neq 0$, we define $\Theta = \arcsin(A_1 \sin(wt_1 + \phi_1)/A_2)$. From the first equation of Equation (13), we have the either $wt_1 + \phi_2 = \Theta$ or $wt_1 + \phi_2 = \pi - \Theta$. We use the second equation from Equation (13) to determine which of these two equations to use. Using this, we can create the piece-wise function

$$z = \begin{cases} A_1 \sin(wt + \phi_1) & \text{if } t < t_1 \\ A_2 \sin(wt + \phi_2) & \text{if } t > t_1. \end{cases} \tag{16}$$

We have two examples illustrated in Figure (4). We plot the graph of position of the platform $z(t)$ with the addition of an applied impulse. A dot is added to imply the point at which the impulse is applied. The parameters for the left figure are $A_1 = 1, w = 2\pi$, $\phi_1 = -\pi/4, I = 0.3$ and $t_1 = 3.1$. The parameters for the right figure are $A_1 = 1, w = 2\pi$, $\phi_1 = -\pi/4, I = 0.3$ and $t_1 = 3.1$. In the left figure, the amplitude of the

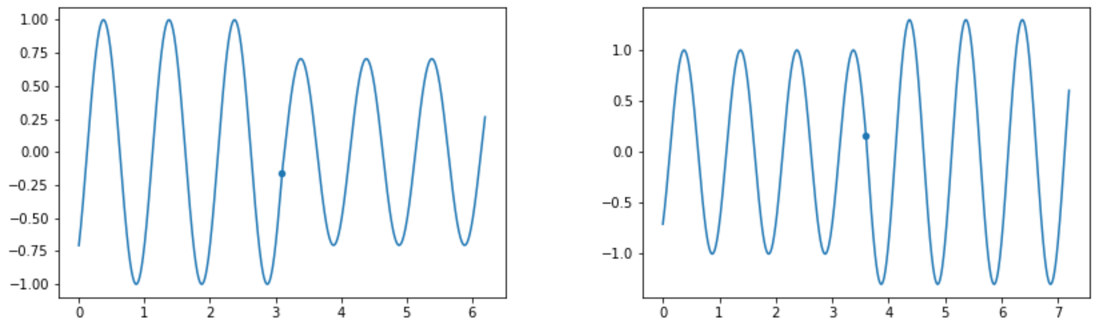

Figure 4: Illustration of the change in the dynamics of the amplitude of the oscillations of the platform when subject to an impulse represented by the solid dot.

oscillations decreased following the initial impulse. This occurred because the impulse was applied at a point in time where the platform was moving upward, and because the impulse applied a downward force, the impulse slowed the platform's motion. In the right figure, the amplitude of the oscillations increased after the impulse was applied. Because the impulse occurred when the platform was moving downward, the direction of the impulse, the force of the impulse is added to the platform's momentum. Thus, the velocity the amplitude of the oscillations increased.

4 The Mathematical Model of a Bridge With Pedestrians Walking in Unison

We model the effects on the dynamics of the platform as it receives an impulse $-J$ as pedestrians walked across it in unison. There is a negative sign in front of J because the impulse exerts a downward force on the platform. Each time a foot hits the platform from a step taken by the pedestrians, there is an impulse. We make the assumption that the pedestrians are making perfectly synchronized, timed steps. We set the constant T to be a positive number to represent the time intervals at which each step will occur. For instance, if $T = 4$, an impulse will be exerted on the platform every 4 units of the selected time unit. We set $t_n = nT$. The platform will be subjected to the impulse $-J$ each time $t = t_n$, for every non-negative integer of n. We use this division of time intervals and the equation from the previous section to have that

$$z = A_n \sin(wt + \phi_n) \quad \text{if } t_{n-1} < t < t_n, \quad \text{where } t_n = nT\,. \tag{17}$$

As we did before in the previous section, we define $I = J/Mw$. With this new measurement of time intervals of the impulses, we can modify Equation (15) to account for repeated step patterns. This leads to the equation for the amplitudes

$$A_{n+1} = \sqrt{A_n^2 - 2A_n I \cos(wt_n + \phi_n) + I^2} \quad \text{and} \quad A_0 = 0\,. \tag{18}$$

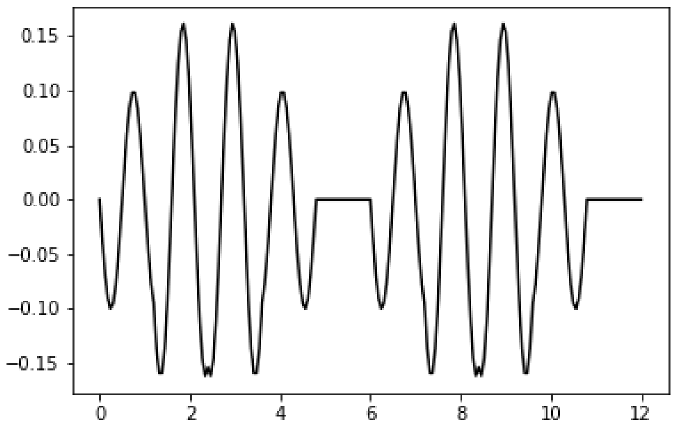

Figure 5: Plot of $z(t)$ where $w = 2\pi, I = 0.2$ and $T = 1.2$.

We set $A_0 = 0$ to indicate that the platform was not oscillating prior to the initial impulse $-J$ from the first step of the pedestrians. We also have the equation $\phi_{n+1} = 0, if A_{n+1} = 0$, otherwise

$$\phi_{n+1} = \begin{cases} \arcsin\left(\frac{A_n}{A_{n+1}} \sin(wt_n + \phi_n)\right) - wt_n & \text{if } A_n \cos(wt_n + \phi_n) - I > 0 \\ \pi - \arcsin\left(\frac{A_n}{A_{n+1}} \sin(wt_n + \phi_n)\right) - wt_n & \text{if } A_n \cos(wt_n + \phi_n) - I < 0\,. \end{cases} \tag{19}$$

In Figure 5, we see a plot for $z(t)$. In this figure, $w = 2\pi, I = 0.2$ and $T = 1.2$. Note that $z(t)$ is a periodic function of of t and thus its amplitude remains bounded. We see that $|z(t)| < 0.16$ for the whole sequence. The period of $z(t)$ is 6. This can also be interpreted by stating that, $z(t+6) = z(t)$ for all values of t.

Observation: If there are positive integers m and k such that $wmT = 2k\pi$ and m does not divide k, then z(t) is periodic with period mT, because the frequencies of the impulses and the oscillations will not align consistently.

In the case of Figure 5, $m = 5$ and $k = 6$, which satisfies the conditions observed because $wmT = 2\pi 5(1.2) = 12\pi$ and $2km = 2(6)\pi = 12\pi$. As a result, Figure 5 corresponds to nonresonant case.

In Figure 6, we again see a plot for $z(t)$. However, in this example, $w = 2\pi, I = 0.2$ and $T = 1$. Note that in this case, $z(t)$ is not a periodic function of t. The amplitude of this platform will continue to grow linearly as time passes and will continue to grow indefinitely. We have that $z(n + \frac{3}{2}) = nI$ for all positive integers of n.

Observation: If there are positive integers k such that $wT = 2k\pi$, then the amplitude of $z(t)$ will grow linearly with time. More precisely, $z \approx -\frac{I}{T} t \sin(wt)$, where the symbol $\approx$ means asymptotically as t becomes larger. The equation is an approximation of $z(t)$, but a fairly accurate one, that will increase in accuracy the larger t is. This example in Figure 6 is a resonant case. This corresponds to the given scenario of soldiers marching in unison to collapse the bridge. The bridge can collapse due to the frequency of the footsteps corresponding to the frequency of the bridge.

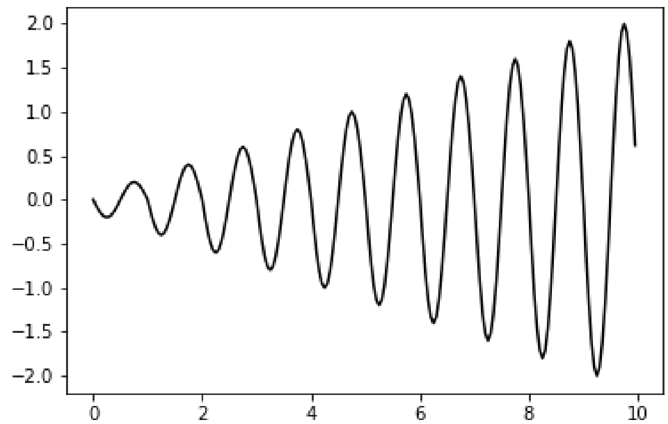

Figure 6: Plot of $z(t)$ where $w = 2\pi, I = 0.2$ and $T = 1$.

5 Discussion

This article discussed the mathematical model of the effects of pedestrians on the dynamics of the bridge's motion as they cross it in unison. The phenomenon displayed in this instance is known as force oscillators. In our analysis of the example, we distinguish a resonant reaction and a nonresonant reaction to specific intervals of footsteps. In the case of a resonant reaction, the amplitude of the oscillations would continue to increase indefinitely, until either the pedestrians cease their footsteps or the bridge breaks and collapses. This defined mathematical model does not consider many external physical factors that would have an additional effect on the bridge, causing it to slightly deviate from the given calculations.

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