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Discovering Companion Pell Numbers

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1 Introduction

In an article from the last issue of *Parabola*, Randell Heyman [1] used the Binomial Theorem to show that, for each natural number n,

$$c_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$$

is an integer. It is natural to ask whether we can define the sequence using a simpler formula or a recursive relation. It turns out there exists a nice and simple recursive relation for this sequence.

2 Finding the recursive relation

Proposition 1. *The sequence $c_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$ satisfies the following recursive relation*

$$c_n = 2c_{n-1} + c_{n-2}$$

with the initial conditions $c_1 = 2$ and $c_2 = 6$.

It is quick and easy to prove this proposition but instead of presenting a proof, let us instead show how one might find the recursive relation in the first place.

For convenience, we denote

$$\begin{aligned} u &= \sqrt{2}+1 \\ v &= \sqrt{2}-1 \end{aligned}$$

so that $uv = 1$; then c_n can be expressed as

$$c_n = u^n + (-1)^n v^n$$

or, equivalently, as

$$c_n = \begin{cases} u^n - v^n & , \ n \text{ is odd} \\ u^n + v^n & , \ n \text{ is even.} \end{cases}$$

We now search for recursive relations for the sequence c_n for when n is odd and when n is even.

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Case 1: n is odd.

We first write $n = 2k + 1$ for some positive integer k. Then c_n can be factored out and simplified by the identities $uv = 1$ and $u - v = 2$:

$$\begin{aligned} c_n &= u^{2k+1} - v^{2k+1} \\ &= (u-v)(u^{2k} + u^{2k-1}v + \cdots + u^k v^k + u^{k-1}v^{k+1} + \cdots + v^{2k}) \\ &= 2(u^{2k} + u^{2k-2} + \cdots + u^2 + 1 + v^2 + v^4 + \cdots + v^{2k})\,. \end{aligned}$$

Since $n - 2$ is also an odd number, the term c_{n-2} can be written as

$$c_{n-2} = 2(u^{2k-2} + \cdots + u^2 + 1 + v^2 + v^4 + \cdots + v^{2k-2})\,.$$

Therefore,

$$c_n - c_{n-2} = 2(u^{2k} + v^{2k}) = 2c_{n-1}\,.$$

We therefore find the recursive relation as follows when n is odd:

$$c_n = 2c_{n-1} + c_{n-2}\,.$$

Case 2: n is even

Is the recursive relation for Case 2 the same as the relation for Case 1? Let's calculate the difference $c_n - c_{n-2}$:

$$\begin{aligned} c_n - c_{n-2} &= (u^n + v^n) - (u^{n-2} + v^{n-2}) \\ &= (u^n - u^{n-2}) + (v^n - v^{n-2}) \\ &= u^{n-2}(u^2 - 1) + v^{n-2}(v^2 - 1) \\ &= u^{n-2}(2 + 2\sqrt{2}) + v^{n-2}(2 - 2\sqrt{2}) \\ &= 2(u^{n-2} + v^{n-2}) + 2\sqrt{2}(u^{n-2} - v^{n-2}) \\ &= 2c_{n-2} + (u+v)(u^{n-2} - v^{n-2}) \\ &= 2c_{n-2} + u^{n-1} - v^{n-1} + u^{n-2}v - uv^{n-2} \\ &= 2c_{n-2} + c_{n-1} + u^{n-3} - v^{n-3} \\ &= 2c_{n-2} + c_{n-1} + c_{n-3}\,. \end{aligned}$$

Since both $n - 1$ and $n - 3$ are odd numbers, we can apply the relation from Case 1 to c_{n-1} and c_{n-3}. Then we have $c_{n-1} = 2c_{n-2} + c_{n-3}$ and the difference is

$$\begin{aligned} c_n - c_{n-2} &= 2c_{n-2} + c_{n-1} + c_{n-3} \\ &= (2c_{n-2} + c_{n-3}) + c_{n-1} \\ &= c_{n-1} + c_{n-1} \\ &= 2c_{n-1} \end{aligned}$$

which is the same as the recursive relation in Case 1.
Now that we have found the relation, we just need to find the initial conditions for it:

$$\begin{aligned} c_1 &= (1 + \sqrt{2}) + (1 - \sqrt{2}) = 2 \\ c_2 &= (1 + \sqrt{2})^2 + (1 - \sqrt{2})^2 = 6\,. \end{aligned}$$

3 Companion Pell numbers

With the recursive relation, we could calculate the elements of the sequence by hand. However, it is convenient to use computer programmings to find the terms. The following MATLAB [2] code lists the first n elements of the sequence:

```
A=[2, 6];
n=input('Enter the number of terms you want: ');
for i=3:n
    A(i)=2*A(i-1)+A(i-2);
end
```

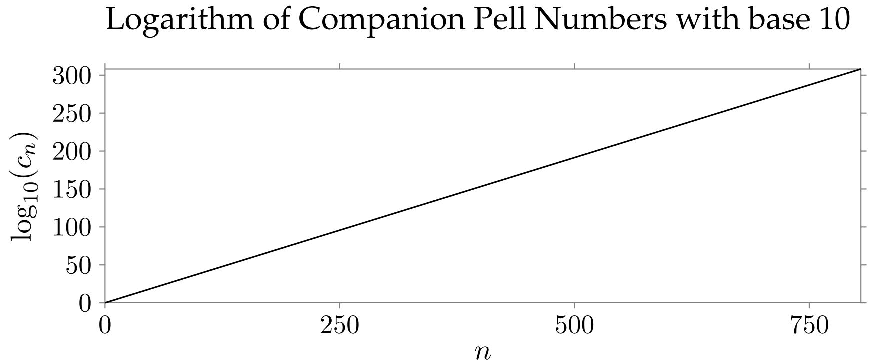
The first ten terms of the sequence are

$$2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, \ldots .$$

These numbers are known as Companion Pell numbers [3]. The elements of the sequence grow so fast that MATLAB can only calculate up to the 805^{th} companion Pell number which is roughly 1.3628×10^{308}. MATLAB will output "INF", which means infinity, for all terms from the 806^{th} term onwards. In order to observe how fast these terms are increasing, we plot the first 805 points $(n, \log_{10} c_n)$ with the following MATLAB code:

```
Seq=[2, 6];
for i=3:805
    Seq(i)=2*Seq(i-1)+Seq(i-2);
end
plot(1:805, log10(Seq))
xlabel('n')
ylabel('log_{10}(c_n)')
title('Logarithm of Companion Pell Numbers with base 10')
```

The output is the figure below, here tidied up slightly.



From the figure, we guess that $y = \log c_n$ approaches a line as n grows large and we can find the approximate value of the n^{th} companion Pell number. If there is such an asymptote for $y = \log c_n$, then what's the slope of the asymptote? We can find the slope by calculating the limit of the ratio $\frac{\log c_n}{n}$.

Proposition 2. *The n^{th} companion Pell number approaches $10^{n(\log\sqrt{2}+1)}$ as n grows large.*

Proof. Let's first rewrite $\log c_n$ by multiplying $(1+\sqrt{2})^n$ for both the nominator and the denominator:

$$\begin{aligned}\log c_n &= \log\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right)\\ &= \log \frac{(1+\sqrt{2})^{2n} + (-1)^n}{(1+\sqrt{2})^n}\\ &= \log\left((\sqrt{2}+1)^n + \frac{(-1)^n}{(\sqrt{2}+1)^n}\right).\end{aligned}$$

As n goes to infinity, $(\sqrt{2}+1)^n$ goes to infinity and $\dfrac{(-1)^n}{(\sqrt{2}+1)^n}$ goes to zero.

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\frac{\log c_n}{n} &= \lim_{n\to\infty}\frac{\log(\sqrt{2}+1)^n}{n}\\ &= \lim_{n\to\infty}\frac{n\log(\sqrt{2}+1)}{n}\\ &= \log(\sqrt{2}+1).\end{aligned}$$

We conclude that $\log(c_n)$ approaches $n\log(\sqrt{2}+1)$ as n grows large, and the line $y = n\log(\sqrt{2}+1)$ is an asymptote of $y = \log c_n$. The n^{th} companion Pell number c_n approaches $10^{n(\log\sqrt{2}+1)}$ as n grows large. $\square$

References

[1] Randel Heyman, Strange irrational powers, *Parabola* **56 (3)** (2020), https://www.parabola.unsw.edu.au/2020-2029/volume-56-2020/issue-3/article/strange-irrational-powers last accessed on 03-07-2021.

[2] Matlab from Mathworks, https://www.mathworks.com, last accessed on 03-07-2021.

[3] Companion Pell numbers, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org/A002203, last accessed on 03-07-2021.