Parabola Volume 57, Issue 1 (2021)

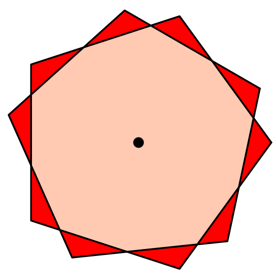
Problems 1641–1650

Parabola would like to thank Toyesh Prakash Sharma for contributing Problem 1641, Jason Zimba for Problem 1642, John Winkelman for Problem 1643 and Jim Franklin for Problem 1644.

Q1641 Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$. Prove that

$$\left(\frac{16}{a^2} - 1\right)\left(\frac{16}{b^2} - 1\right)\left(\frac{16}{c^2} - 1\right)\left(\frac{16}{d^2} - 1\right) \geq 15^4 .$$

Q1642 A regular n–gon is rotated by some angle about its centre O and the result is superimposed upon the original; the diagram illustrates the situation for $n = 5$.



Let A_0 be the area of the original polygon and P_0 its perimeter. Let A be the area common to both polygons (light red in the figure) and P the perimeter of the combined polygons (the whole coloured region in the figure). Prove that

$$\frac{P}{P_0} + \frac{A}{A_0} = 2 .$$

Q1643 A positive integer with k digits $d_0 d_1 \cdots d_{k-1}$ in base 10 is called a Geezer number if the digits consist of exactly d_0 zeros, exactly d_1 ones, exactly d_2 twos and so on. The number of digits is at most 10. For example, 2020 is a Geezer number since the digits $0, 1, 2, 3$ occur $2, 0, 2, 0$ times. We do not allow the first digit of a positive integer to be zero. Prove that in a k–digit Geezer number

(a) the sum of the digits is k;

(b) the digits $d_3, d_4, \ldots, d_{k-1}$ cannot be greater than 1.

Q1644 Of the students in a maths class, the proportion who read *Parabola* is 66%, *to the nearest percent*. What is the smallest possible number of students in the class?

Q1645 A row of boxes contains m zeros, followed by the numbers $1, 2, \ldots, n$. There is a row of empty boxes below it. We give an example with $m = 3$ and $n = 5$.

0	0	0	1	2	3	4	5

We want to write the same numbers in the second row in such a way that no column contains the same number twice. Determine the number of ways of doing this

(a) if $n = m$;

(b) if $n = m + 1$;

(c) if $n = m + 2$.

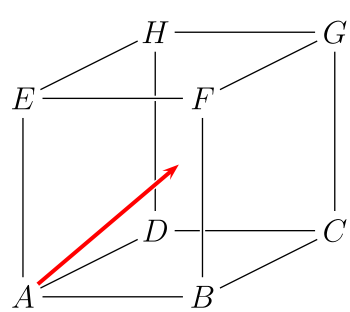
Q1646 Triangle ABC has a right angle at B, and D is a point on the hypotenuse AC. The perpendicular to AC at D intersects AB at E, and we draw the line EC.

Use this diagram to prove the "cosine of a sum" formula

$$\cos(x + y) = \cos x \cos y - \sin x \sin y\,.$$

Q1647 A monk visits t temples and burns a number of incense sticks, the same number at each temple. The temples are located on different islands in a magic lake and he visits them by boat. The lake doubles the number of sticks he holds each time he reaches an island. At the end of the day he has burnt all his incense sticks. What is the smallest number of incense sticks that the monk could initially have had?

Q1648 A rectangular box $ABCDEFGH$ has side lengths $AB = 19$, $AD = 20$, $AE = 21$, and has a small aperture at each of the vertices. A particle P is projected from A to the interior of the box along the line $x = y = z$. For instance, if the origin is at $A = (0, 0, 0)$ and the x, y, z axes are along AB, AD, AE respectively, the particle P will first rebound from the wall $BCGF$ at the coordinate $(19, 19, 19)$. Which vertex will P emerge from eventually?



Q1649

(a) Given positive real numbers p, q, find all real numbers x, y with $x > y \geq 0$ such that

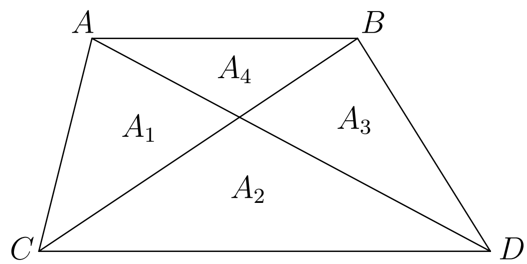
$$\sqrt{x} - \sqrt{y} = p \quad \text{and} \quad x - \sqrt{xy} + y = q\,.$$

(b) Given positive real numbers s, t, consider the simultaneous equations

$$x^2 + y^2 = s \quad \text{and} \quad 2x\sqrt{xy} - 3xy + 2y\sqrt{xy} = t\,.$$

Show that the equations have no solution if $s < 2t$; and find all solutions $x > y \geq 0$ if $s \geq 2t$.

Q1650 A trapezium $ABDC$ has AB parallel to CD. The diagonals AD and BC divide the trapezium into four triangles with areas A_1, A_2, A_3, A_4 as shown in the diagram.



(a) Prove that $A_1A_3 = A_2A_4$.

(b) Deduce that the area of the trapezium is at least $4A_1$.
What more can you say about the trapezium if its area is $4A_1$?