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Mathematics and Music: finding and characterizing equally tempered scales using continued fractions approximations

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1 Introduction

Music predates modern humans, and musical instruments were most likely built at the same time as the first tools: from the rocks banged together to produce rhythmical sounds, to animal skins stretched into drums and animal bones with holes drilled into them to create flute-like instruments. The fascination for music survived through the ages and with each generation it grew stronger as the instruments and the sounds they produced became more diverse and complex. As the science progressed, many subjects were applied and sometimes developed to study musical instruments and musical theory: algebra, geometry, physics, differential equations, fluid mechanics and so on.

In this paper we look at musical scales, particularly equally tempered scales, which are the foundation of modern Western music. We describe the idea behind the most common number of notes in the scale (12 notes in an octave) and look at ways of developing scales with a higher number of notes. In this context we use mathematical approximations, in particular continued fractions, to approximate irrational numbers with optimal fractions. We also develop an algorithm to classify musical scales with any number of notes to an octave based on their consonance.

2 Pythagorean scales

An early record of a scientific study of music and musical instruments comes in the form of a legend from around 500BC when Pythagoras observed that the pitch of the sound produced by a string is related to its length: halve the length of the string and you raise the pitch by an octave, two-thirds the original length raises the pitch by a fifth, and so on [3]. Some of the sounds produced by these strings were classified as pleasant when played together (consonant) while others were not (dissonant). In particular, when the ratios between lengths of the strings involved small integers (like

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2:1 or 3:2), the sounds were more consonant than those of larger integers or random numbers.

The story goes on to say that Pythagoras used the two most consonant musical intervals (the octave and the fifth) to build the first scale [8]. The idea behind such a scale is to start with a reference octave, say C4-C5 and add to the first note (C4) a perfect fifth interval to discover the next note in the scale (G4). The frequency ratio between G4 and C4 is therefore 3:2 (see Table 1 below). Adding another perfect fifth interval to G4, we obtain a frequency ratio of 9:4 (with respect to C4) which is outside of our octave. Reducing this interval by an octave (by dividing this frequency ratio by 2), we obtain a ratio of 9:8 which is the third note in the scale (D4). Continuing this algorithm we can discover all the notes in the scale (see Table 1). For simplicity we have listed the ratios of the main 7 notes and omitted them for the sharps. Note that, in this system, adding/subtracting intervals amounts to multiplying/dividing frequency ratios, which is why it is commonly referred to as a logarithmic scale (more about this will be discussed later).

Note	C4	D4	E4	F4	G4	A4	B4	C5
Ratio	1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Table 1: One octave of the Pythagorean scale of C major and the corresponding frequency ratios for the 7 main notes in the scale

The Pythagorean scale is based on the assumption that going up 12 fifths and coming down 7 octaves will return you exactly to the same starting note, or, in other words, that

$$2^{19} \approx 3^{12}\,. \tag{1}$$

These numbers are indeed close; however they are not equal to each other and this assumption leads to certain problems specific to the Pythagorean scales. Among the issues are dissonant musical intervals, like the third, which are supposed to be consonant. Other scales were later developed to alleviate some of these problems, for example the just intonation, meantone temperament and equal temperament scales [1].

Before discussing other scale systems, we look at an interesting way of representing frequency ratios in real numbers using logarithms. Each frequency ratios can be transformed via the logarithmic function $\log_2$ to a unique real number which represents that particular musical interval. For example, a frequency ratio of 3:2, representing a fifth, is now an interval of $\log_2 \frac{3}{2} = 0.5849625\ldots$; a frequency ratio of 2:1, representing an octave, is now an interval of $\log_2 \frac{2}{1} = 1$ and so on (see Table 2). Products and fractions of frequency ratios correspond to sums and differences of their real logarithmic representations, which is more intuitive.

One interesting take from this representation is that the length of one octave is equal to 1, while the length of a perfect fifth is the irrational number $\log_2 \frac{3}{2}$. In a scale system based on the Pythagorean approach, we are trying to add multiples of this irrational number to get an integer number; this is impossible and therefore leads to

Note	C4	D4	E4	F4	G4	A4	B4	C5
Real	0	0.16..	0.33..	0.41..	0.58..	0.75..	0.92..	1

Table 2: One octave of the Pythagorean scale of C major and the real number representations of the 7 main notes in the scale.

the problems described above. In the next section, we will look at equally tempered scales which were developed as a compromise aimed at fixing some of these issues present in Pythagorean scales.

3 Equally tempered scales

The idea behind equally tempered scales is very simple: take the interval [0,1], which represents one octave of a scale, and equally split it into 12 sub-intervals. The resulting end-points of this partition are defined as the notes in the scale (see Table 3 below).

Note	C4	D4	E4	F4	G4	A4	B4	C5
Real	0	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{9}{12}$	$\frac{11}{12}$	1

Table 3: One octave of the 12-note equally tempered scale of C major and the real number representations of the 7 main notes in the scale.

The equal temperament scale makes all the tones and semitones equal to each other. It is designed to correct all the issues associated with the Pythagorean scales by distributing the dissonance occurring in different intervals throughout the entire scale. It is obvious that the original intervals like the perfect fifth no longer have frequency ratios represented by small integers. However, the advantage is that these highly consonant intervals are going to sound the same (and still relatively consonant) in any key and any pitch. Because of its homogeneity and universality, all modern instruments are tuned in this scale and all Western, modern music is based on musical intervals relative to this equally tempered scale.

We are going to take a closer look at the original interval that we discussed, the perfect fifth, in this new scale. Notice from Table 3 that the perfect fifth has a length of $\frac{7}{12}$ units, and this can be understood in the sense that, in the C major scale, the note G is the 7-th note out of a total of 12 notes. It is also an approximation of the original interval representing the perfect fifth

$$\log_2\left(\frac{3}{2}\right) \approx \frac{7}{12}\,.$$

This points to the idea that finding rational approximations to the irrational number $\log_2\left(\frac{3}{2}\right)$ could lead to the discovery of new scales, where the denominator of the rational approximation represents the total number of notes in the scale, and the numerator represents the position of the perfect fifth relative to the first note in the scale. Theoretically finding these approximations can be done for any interval in the scale, but logically one wants to do it for the most consonant musical interval (the perfect fifth) in order to control this approximation and to discover a scale in which the perfect fifth interval has a high degree of consonance.

There are many methods for approximating irrational numbers with rational fractions. For example, a simple, brute, method would be to just truncate the infinite number of decimals after the decimal point, e.g.,

$$\log_2\left(\frac{3}{2}\right) = 0.584962500721.... \approx 0.58 = \frac{29}{50}\,.$$

This approximation would indicate the possibility of using a scale with 50 notes to an octave with the perfect fifth located at note 29 with respect to the first note. There are some obvious issues with this method: for example, large numerators (and therefore a large number of notes to an octave in the scale); numerators increasing very quickly from one approximation to the next (and therefore providing a extremely small number of useful examples); and the lack of flexibility in the number of notes to an octave (always multiples of 10 or their divisors).

In this paper, we focus exclusively on the method of continued fractions to produce rational approximations to irrational numbers. The method and examples are discussed in the next section.

4 Continued fractions

A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where a_i are integers and usually positive for any $i \geq 0$. This expression can go on to infinity or it can stop after a finite number of iterations. Every real number has a continued fraction representation and it is not difficult to see that irrational numbers yield an infinite such expression while the rational numbers have finite representations. For example,

$$\pi = 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$

and

$$\frac{22}{7} = 3 + \frac{1}{7}\,.$$

To get a good approximation for irrational numbers, one has to stop just before a large value of a_i. In the example above, stopping the continued fraction for π just before 15

produces the famous approximation

$$\pi \approx \frac{22}{7}\,.$$

Another possibility in the same example would be to truncate the continued fraction just before 292 (another large a_i) would produce the approximation

$$\pi \approx \frac{355}{113}$$

and so on. Both approximations are very good rational approximations of π; however, the first is more useful as it contains smaller numbers which are easier to handle in mathematical calculations.

The many properties and applications of continued fractions are described in classical books like [4, 5] and a comprehensive review of them is beyond the scope of this article.[5]

We instead focus on applying this method to discover equally tempered scales with a consonant perfect fifth interval. As described above, we will use continued fractions to approximate the irrational number $\log_2\left(\frac{3}{2}\right)$ to obtain

$$\log_2\left(\frac{3}{2}\right) = 0 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{23+\ldots}}}}}}}}}$$

and so on. Some notable truncations of this expansion are as follows. If we stop just before $a_5 = 3$, then we obtain the approximation

$$\log_2\left(\frac{3}{2}\right) \approx \frac{7}{12}$$

which indicates an equally tempered scale with 12 notes to the octave and a perfect fifth interval located at the 7th note. This is the modern Western scale currently used today in all music compositions. If we stop just before the $a_7 = 5$, then we obtain the approximation

$$\log_2\left(\frac{3}{2}\right) \approx \frac{31}{53}$$

which indicates an equally tempered scale with 53 notes to the octave and a perfect fifth interval located at the 31st note. This scale was well known in the 18th century [2] when a generalized keyboard was build with the capability of playing all 53 notes in an octave. The next logical truncation of the continued fractions would be right before $a_9 = 23$, which would give the approximation

$$\log_2\left(\frac{3}{2}\right) \approx \frac{389}{665}$$

[5]For more information on continued fractions, read the excellent *Parabola* article Integer Points, Conics and Continued Fractions by Peter Brown; see also the earlier *Parabola* article here.

indicating an equally tempered scale with 665 notes to the octave and a perfect fifth interval located at the 389th note. However, the benefits of getting a better approximation to the fifth interval (and therefore a more consonant fifth) are outweighed by the large number of tones in the scale, with consecutive tones being indistinguishable from each other and the impossibility of building instruments capable of reproducing all these notes.

One way of studying the accuracy of the continued fractions approximations is by using the Hurwitz Theorem [5], which states that, for every irrational number ξ, there are infinitely many relatively prime integers m and n such that

$$\left|\xi - \frac{m}{n}\right| < \frac{1}{\sqrt{5}n^2}\,.$$

The theorem provides a way of testing the quality of rational approximations to irrational numbers: a good approximation $\frac{m}{n}$ to an irrational ξ will be within $\frac{1}{\sqrt{5}n^2}$ distance of it. Following [6], if we denote the approximation error by E and the Hurwitz bound by M, then the size of the ratio E/M provides an indication of the quality of the approximation: the smaller the ratio, the higher the quality of the approximation. These characteristics are captured in Table 4 for our previous example, the irrational number π, and for $\log_2\left(\frac{3}{2}\right)$.

Irrational number	Approximation	Error E	Hurwitz bound M	E/M
π	22/7	0.001264489267	0.009126808071	0.139
	355/113	0.000000266764	0.000035023384	0.008
$\log_2\left(\frac{3}{2}\right)$	7/12	0.001629167388	0.003105649969	0.525
	31/53	0.000056840343	0.000159207403	0.357
	389/665	0.000000094706	0.000001011280	0.094

Table 4: Hurwitz characteristics for different continued fractions approximations for the irrational numbers π and $\log_2\left(\frac{3}{2}\right)$

The small values in the last column in Table 4 above are an indication of the high quality of the rational approximations. It is also a confirmation that the method of truncating a continued fraction just before a large value of a_i leads to a good rational approximation of the particular irrational number.

5 Consonance score

To differentiate between all the equally tempered scales described above, we have developed an algorithmic approach to characterizing them based on their level of consonance. It has long been known that musical intervals like the fifth, the third etc. have higher degree of consonance than other random intervals. Plomp and Levelt [7] ranked six of the most consonant musical intervals as seen in Figure 1 .

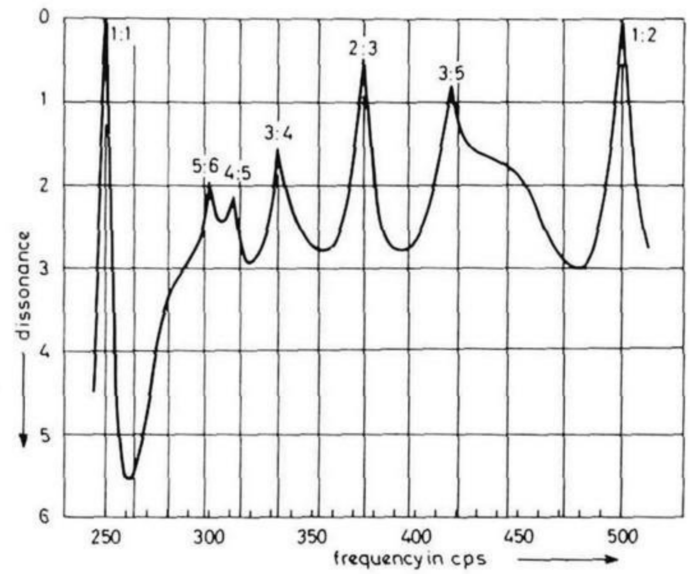

Figure 1: A dissonance curve for a 6-partial complex tone (from [7]). Vertical lines represent the location of the musical notes in a 12-note equally tempered scale.

Starting with any equally tempered scale (containing between 1 and 100 notes to an octave), we assign a consonance score based on the differences between the Pythagorean notes (the peaks of the curve in Figure 1) and the closest note from the equally tempered scale (vertical lines in Figure 1). The reciprocal of each difference is weighted according to the relative dissonance scores of the six ratios in Figure 1, where a lower dissonance score equates to a higher consonance and relative weight. These are then added together to obtain the consonance score for that scale. Note that the vertical lines in Figure 1 will change based on the number of notes in the equally tempered scale that we are scoring, while the peaks of the consonance curve will not. The results of this test are shown in Figure 2 .

Surprisingly, the highest score is assigned to the scale with 53 notes to the octave indicating a scale with highly consonant musical intervals. Other notable scales are 19, 38, 41, 75, 94 and others. The 12-note scale does not get an exceptional score in this system; however it is worth mentioning that it is the scale with the lowest number of notes where a consonance peak occurs.

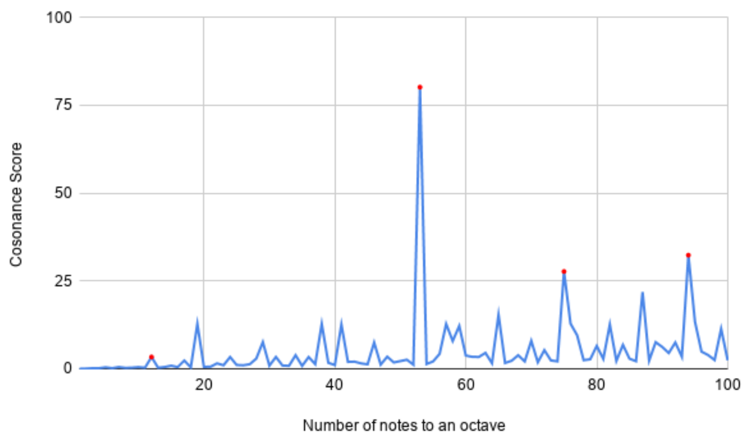

Figure 2: Consonance scores for all equally tempered scales with a number of notes up to 100 in an octave. The red dots indicate the score for the 12-note, 53-note, 75-note and 94-note scales.

6 Conclusions

In this paper we looked at equally tempered musical scales, and in particular at the reasons for having scales with 12-notes in an octave. We outlined methods for developing equally tempered scales with a higher number of notes in one octave based on continued fractions approximations. We also described several approaches for ranking the scales based on the consonance of the classical musical intervals like the perfect fifth.

Besides the interesting results described in this paper, the work had a highly educational value for the authors, exposing them to subjects from algebra, approximations theory, musical scales and computer programming. Although mathematics and music seem at opposite sides of the scientific spectrum, it is refreshing to combine the rigour of mathematics with the flexibility and creativity of arts.

References

[1] J.M. Barbour, *Tuning and Temperament: A Historical Survey*, Michigan State College Press, E. Lansing, 1951.

[2] R. Bosanquet, *Musical Intervals and Temperaments*, Macmillan and Co. 1876.

[3] *The Cambridge History of Western Music Theory*, Cambridge University Press, 2002.

[4] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Fifth edition, 1980.

[5] L.-K. Hua, *Introduction to Number Theory*, Springer-Verlag, Berlin/New York, 1982.

[6] T. Phillips, *The Most Irrational Number*, American Mathematical Society Feature Column, August 1999.

[7] R. Plomp and W.J.M. Levelt, Tonal consonance and critical bandwidth, *JASA* **38** (1965), 548–560.

[8] W.A. Sethares, *Tuning, Timbre, Spectrum, Scale*, Springer 2005.