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On infinite limits involving square roots with quadratic radicands

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1 Introduction

A type of $\infty - \infty$ form limits frequently appearing in calculus exercises (see, e.g., [1, Exercises 21−22 in Problem Set 1.5]) is that which involves the difference of two square root expressions with quadratic radicands, such as

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-\sqrt{x^2+4x}\right).$$

A standard technique for computing such limits is to multiply and divide by the conjugate of the expression:

$$\begin{aligned}&\lim_{x\to\infty}\left(\sqrt{x^2+6x}-\sqrt{x^2+4x}\right)\frac{\sqrt{x^2+6x}+\sqrt{x^2+4x}}{\sqrt{x^2+6x}+\sqrt{x^2+4x}}\\ &=\lim_{x\to\infty}\frac{(x^2+6x)-(x^2+4x)}{\sqrt{x^2+6x}+\sqrt{x^2+4x}}\\ &=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+6x}+\sqrt{x^2+4x}}.\end{aligned}$$

Dividing the numerator and the denominator of the last expression by x, one obtains

$$\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{6}{x}}+\sqrt{1+\frac{4}{x}}}=\frac{2}{\sqrt{1+6\lim\limits_{x\to\infty}\frac{1}{x}}+\sqrt{1+4\lim\limits_{x\to\infty}\frac{1}{x}}}=\frac{2}{\sqrt{1+6\cdot 0}+\sqrt{1+4\cdot 0}}=1.$$

Using the same technique, it is not difficult to show that, if $A>0$, then

$$\lim_{x\to\infty}\left(\sqrt{Ax^2+Bx+C}-\sqrt{Ax^2+bx+c}\right)=\frac{B-b}{2\sqrt{A}};\qquad(1)$$

and if the leading coefficients of the radicands are not equal, then the limit does not exist.

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In this note, we point out that it is possible to generalise the formula (1) to similar limits involving more than two square roots with quadratic radicands. Let us first illustrate the idea by considering the limit

$$\lim_{x\to\infty}\left(\sqrt{9x^2+x}-\sqrt{4x^2+6x}-\sqrt{x^2+2x}\right).$$

This limit can be computed by splitting the first square root expression as follows:

$$\begin{aligned}\sqrt{9x^2+x}&=\sqrt{9\left(x^2+\frac{1}{9}x\right)}\\&=3\sqrt{x^2+\frac{1}{9}x}\\&=(2+1)\sqrt{x^2+\frac{1}{9}x}\\&=2\sqrt{x^2+\frac{1}{9}x}+\sqrt{x^2+\frac{1}{9}x}\\&=\sqrt{4x^2+\frac{4}{9}x}+\sqrt{x^2+\frac{1}{9}x}\,.\end{aligned}$$

Therefore, using (1) twice,

$$\begin{aligned}&\lim_{x\to\infty}\left(\sqrt{9x^2+x}-\sqrt{4x^2+6x}-\sqrt{x^2+2x}\right)\\&=\lim_{x\to\infty}\left(\sqrt{4x^2+\frac{4}{9}x}-\sqrt{4x^2+6x}\right)+\lim_{x\to\infty}\left(\sqrt{x^2+\frac{1}{9}x}-\sqrt{x^2+2x}\right)\\&=\frac{\frac{4}{9}-6}{2\sqrt{4}}+\frac{\frac{1}{9}-2}{2\sqrt{1}}\\&=-\frac{7}{3}\,.\end{aligned}$$

This technique can be used to prove the following general formula.

Proposition. *Let* $n\in\mathbb{N}$*, and let* $A,a_1,\dots,a_n>0$ *satisfy*

$$\sqrt{A}=\sum_{i=1}^{n}\sqrt{a_i}\,. \tag{2}$$

Then

$$\lim_{x\to\infty}\left(\sqrt{Ax^2+Bx+C}-\sum_{i=1}^{n}\sqrt{a_ix^2+b_ix+c_i}\right)=\frac{B}{2\sqrt{A}}-\sum_{i=1}^{n}\frac{b_i}{2\sqrt{a_i}}\,. \tag{3}$$

Proof. Using (2), one obtains

$$\sqrt{Ax^2+Bx+C}=\sqrt{A}\sqrt{x^2+\frac{B}{A}x+\frac{C}{A}}=\sum_{i=1}^{n}\sqrt{a_ix^2+\frac{Ba_i}{A}x+\frac{Ca_i}{A}}\,.$$

Substituting this into the limit on the left-hand side of (3), merging the summations, interchanging the summation and the limit, and using (1) and (2) again, one shows that the limit is equal to

$$\begin{aligned}
\sum_{i=1}^{n} \lim_{x\to\infty} \left(\sqrt{a_i x^2 + \frac{Ba_i}{A}x + \frac{Ca_i}{A}} - \sqrt{a_i x^2 + b_i x + c_i} \right) &= \sum_{i=1}^{n} \frac{\frac{Ba_i}{A} - b_i}{2\sqrt{a_i}} \\
&= \frac{B}{2A}\sum_{i=1}^{n} \sqrt{a_i} - \sum_{i=1}^{n} \frac{b_i}{2\sqrt{a_i}} \\
&= \frac{B}{2\sqrt{A}} - \sum_{i=1}^{n} \frac{b_i}{2\sqrt{a_i}}\,,
\end{aligned}$$

as desired. □

Since limits of this type are ubiquitous, the authors hope that this article could give some inspiration to enrich the materials of calculus or similar mathematical modules.

References

[1] D. Varberg, E. Purcell, and S. Rigdon, *Calculus*, 9^{th} edition, Pearson, New Jersey, 2007.