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Odd perfect numbers

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1 Introduction

Write down all the divisors of 6 except 6 itself: 1, 2, and 3. Add all these divisors. You will come back to the number 6. Interesting, right? Take 28. Do the same thing. You will come back to 28. This is not magic. In fact, 6 and 28 are special numbers. They are called *perfect numbers*. These are the positive integers n that are equal to the sum of their positive divisors d, excluding the number n itself. Since $6 = 1 + 2 + 3$ is the sum of its proper divisors 1, 2, and 3, the number 6 is a perfect number. Other perfect numbers are 28, 496, and 8128.

Euclid was able to demonstrate that if p and $2^p - 1$ are prime numbers, then the number $2^{p-1} \times (2^p - 1)$ is a perfect number. This formula generates only even perfect numbers. For example, if p is 2, then you get 6, and if p is 3, then you find 28. But Euclid couldn't determine whether the set of even perfect numbers is infinite or not.

Only even perfect numbers have been discovered to this day. However, there is no known reason why odd perfect numbers could not exist. Indeed, the Odd Perfect Number Conjecture is one of the oldest unsolved problems in Mathematics: is it true that no odd perfect number exist? This question has intrigued many mathematicians around the world for centuries.

Can you find an odd perfect number or can you prove that it does not exist?

Before you think about this problem, let me introduce to you some necessary (but not sufficient) requirements for the existence of an odd perfect number. These requirements can be found at [1] and many other sources, including many introductory books on number theory. I hope that this introduction might inspire you to read more about perfect numbers.

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2 Necessary requirements for a number to be perfect

Define $\sigma : \mathbb{N} \to \mathbb{N}_0$ to be the function $\sigma(n)$ that, for each positive integer n, equals the sum of all divisors of n, including n itself. Note that a positive integer is perfect exactly when $\sigma(n) = 2n$.

Proposition 1. *Let $n \geq 2$ be a positive integer and write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where $p_1, \ldots, p_k$ are prime numbers and $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$. Then*

$$\sigma(n) = (1 + p_1 + \cdots + p_1^{\alpha_1}) \times \cdots \times (1 + p_k + \cdots + p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \times \cdots \times \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} .$$

Furthermore, σ is multiplicative; that is,

$$\sigma(mn) = \sigma(m)\sigma(n)$$

whenever m and n are co-prime positive integers.

Proof.

$$\sigma(n) = \sum_{d|n} d = \sum_{0 \leq h_1 \leq \alpha_1} \cdots \sum_{0 \leq h_k \leq \alpha_k} p_1^{h_1} \cdots p_k^{h_k} = \sum_{0 \leq h_1 \leq \alpha_1} p_1^{h_1} \sum_{0 \leq h_2 \leq \alpha_2} p_2^{h_2} \cdots \sum_{0 \leq h_k \leq \alpha_k} p_k^{h_k} .$$

Each of the sums above is an geometric sum and therefore is equal to

$$\sum_{0 \leq h_i \leq \alpha_i} p_i^{h_i} = 1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i} = \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} ,$$

so

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \times \cdots \times \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} ,$$

as claimed. Now, suppose that m is a positive integer that is coprime with n, and factorise m into primes q_i: $m = q_1^{\beta_1} \cdots q_\ell^{\beta_\ell}$. Then

$$\begin{aligned} \sigma(mn) &= \sigma(p_1^{\alpha_1} \cdots p_k^{\alpha_k} q_1^{\beta_1} \cdots q_\ell^{\beta_\ell}) \\ &= \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \times \cdots \times \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} \times \frac{q_1^{\beta_1+1} - 1}{q_1 - 1} \times \cdots \times \frac{q_\ell^{\beta_\ell+1} - 1}{q_\ell - 1} \\ &= \sigma(m)\sigma(n) . \end{aligned}$$

□

Theorem 2. *If n is an odd perfect number, then*

$$n = p^r s^2$$

for some prime number p and positive integers r and s such that $p \equiv r \equiv 1 \pmod 4$.

Proof. Assume that n is an odd perfect number and factorise n as $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Then $\sigma(n) = 2n$. On the other hand, according to Proposition 1,

$$\sigma(n) = \sigma(p_1^{\alpha_1}) \cdots \sigma(p_k^{\alpha_k}) = 2n\,.$$

Therefore, $\sigma(p_1^{\alpha_1}) \cdots \sigma(p_k^{\alpha_k})$ is even but is not divisible by 4.

This is possible if and only if one of $\sigma(p_i^{\alpha_i})$ is divisible by 2 and the others are odd. Without loss of generality, suppose that

$$\sigma(p_1^{\alpha_1}) \equiv 2 \pmod 4$$

and that $\sigma(p_2^{\alpha_2}), \ldots, \sigma(p_k^{\alpha_k})$ are odd.

Since p_1 is odd, then either $p_1 \equiv 1 \pmod 4$ or $p_1 \equiv -1 \pmod 4$. Suppose that $p_1 \equiv -1 \pmod 4$. Then

$$\sigma(p_1^{\alpha_1}) = 1 + p_1 + p_1^2 + \cdots + p_k^{\alpha_1} \equiv 1 - 1 + 1 - \cdots + (-1)^{\alpha_1} \pmod 4\,.$$

If α_1 is odd, then $\sigma(p_1^{\alpha_1}) \equiv 0 \pmod 4$; otherwise, $\sigma(p_1^{\alpha_1}) \equiv 1 \pmod 4$. This is a contradiction since $\sigma(p_1^{\alpha_1}) \equiv 2 \pmod 4$.

Therefore, $p_1 \equiv 1 \pmod 4$. Also,

$$\sigma(p_1^{\alpha_1}) = 1 + p_1 + p_1^2 + \cdots + p_k^{\alpha_1} \equiv 1 + 1 + 1 + \cdots + 1 \equiv \alpha_1 + 1 \pmod 4\,.$$

Since $\sigma(p_1^{\alpha_1}) = 2$, it follows that $\alpha_1 \equiv \sigma(p_1^{\alpha_1}) - 1 \equiv 2 - 1 \equiv 1 \pmod 4$.

On the other hand, for $i = 2, \ldots, k$, the prime p_i is odd, so

$$\sigma(p_i^{\alpha_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i} \equiv 1 + 1 + \cdots + 1 \equiv \alpha_i + 1 \pmod 2\,.$$

However, $\sigma(p_i^{\alpha_i}) \equiv 1 \pmod 2$, so $\alpha_i \equiv 0 \pmod 2$; that is, α_i is even.

We have proved that $p_1 \equiv \alpha_1 \equiv 1 \pmod 4$ and that α_i is even for all $i = 2, \ldots, k$. Therefore, we can write

$$n = p^r s^2$$

where prime $p = p_1$ and positive integer $r = \alpha_1$ satisfy $p \equiv r \equiv 1 \pmod 4$ and where $s = p_2^{\alpha_2/2} \cdots p_k^{\alpha_k/2}$. □

Corollary 3. *No odd perfect number is divisible by* 105.

Proof. Assume that n is an odd perfect number that is divisible by $105 = 3 \times 5 \times 7$. We can therefore write n as

$$M = 3^a \times 5^b \times 7^c \times p_4^{\alpha_4} \times \cdots \times p_k^{\alpha_k}$$

for primes $p_4, \ldots, p_k$ and positive integers $a, b, c, \alpha_4, \ldots, \alpha_k$.

Since $3 \equiv 7 \equiv 3 \pmod 4$, Theorem 2 implies that a and c are even; therefore, $a, c \geq 2$. Also, n is perfect, so $\sigma(n) = 2n$. By Theorem 1,

$$\begin{aligned}
2 &= \frac{\sigma(n)}{n} \\
&= \frac{(1+3+\cdots+3^a)(1+5+\cdots+5^b)(1+7+\cdots+7^c)\cdots(1+p_k+\cdots+p_k^{\alpha_k})}{3^a \times 5^b \times 7^c \times \cdots \times p_k^{\alpha_k}} \\
&= (1+3^{-1}+\cdots+3^{-a})(1+5^{-1}+\cdots+5^{-b})(1+7^{-1}+\cdots+7^{-c})\cdots(1+p_k^{-1}+\cdots+p_k^{-\alpha_k}) \\
&\geq (1+3^{-1}+3^{-3})(1+5^{-1})(1+7^{-1}+7^{-2}) \\
&= \frac{494}{245} \\
&> 2\,,
\end{aligned}$$

a contradiction. Therefore, n is not divisible by 105. □

Theorem 4. *An odd perfect number is divisible by at least three different primes.*

To prove this theorem, we will need the following lemma.

Lemma 5. *Let n be an odd perfect number and let $p_1, \ldots, p_k$ be its prime divisors. Then*

$$\frac{p_1}{p_1 - 1} \times \cdots \times \frac{p_k}{p_k - 1} > 2\,.$$

Proof. Write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and note that $\sigma(n) = 2n$. By Theorem 1,

$$2 = \frac{\sigma(n)}{n} = \frac{\frac{p_1^{\alpha_1+1}-1}{p_1-1} \times \cdots \times \frac{p_k^{\alpha_k+1}-1}{p_k-1}}{p_1^{\alpha_1} \cdots p_k^{\alpha_k}} < \frac{\frac{p_1^{\alpha_1+1}}{p_1-1} \times \cdots \times \frac{p_k^{\alpha_k+1}}{p_k-1}}{p_1^{\alpha_1} \cdots p_k^{\alpha_k}} = \frac{p_1}{p_1 - 1} \times \cdots \times \frac{p_k}{p_k - 1}\,.$$

□

Now, let's prove Theorem 4.

Proof. Let n be an odd perfect number and assume that n is divisible only by a single prime number; that is, $n = p^\alpha$ for some prime $p \geq 3$ and an integer α. Then $3p \geq 2p + 3$, so $3(p-1) \geq 2p$. By Lemma 5,

$$\frac{3}{2} \geq \frac{p}{p-1} > 2\,,$$

a contradiction. Assume then than n is divisible by exactly distinct primes; that is, $n = p^\alpha q^\beta$ for primes $p \geq 3$ and $q \geq 5$ and positive integers α and β. Then $\frac{p}{p-1} \leq \frac{3}{2}$ and, similarly, $\frac{q}{q-1} \leq \frac{5}{4}$. But then Lemma 5 implies that

$$2 <\leq \frac{15}{8} = \frac{3}{2} \times \frac{5}{4} \geq \frac{p}{p-1} \times \frac{q}{q-1} > 2\,,$$

a contradiction. It follows that n is divisible by at least 3 distinct prime numbers. □

References

[1] Wikipedia, *Perfect number,*
https://en.wikipedia.org/wiki/Perfect_number,
last accessed on 15-08-2021.