Parabola Volume 57, Issue 2 (2021)

A generalised recurrence relation for irrational powers

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1 Introduction

In previous issues of *Parabola*, Randell Heyman [1] showed that

$$c_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$$

is an integer for each natural number n and Xiaoyan Hu [2] derived a recursion relation for this sequence. This article extends upon these results, by providing another method to arrive at the recursive relation. Indeed, we find the recursive relation for the more general sequence

$$s_n = (a+\sqrt{b})^n + (a-\sqrt{b})^n \qquad \text{for} \qquad a, b \in \mathbb{N}\,.$$

2 Recursive relations

To find the recursive relation for

$$s_n = (a+\sqrt{b})^n + (a-\sqrt{b})^n \qquad \text{for} \quad a, b \in \mathbb{N}\,.$$

we can first find the monic quadratic polynomial with roots $\alpha_\pm = a \pm \sqrt{b}$:

$$\begin{aligned}(x-\alpha_-)(a-\alpha_+) &= \big(x-(a-\sqrt{b})\big)\big((x-(a+\sqrt{b})\big)\\ &= x^2 - \big((a-\sqrt{b})+(a+\sqrt{b})\big)x + (a-\sqrt{b})(a+\sqrt{b})\\ &= x^2 - 2ax + (a^2-b)\,.\end{aligned}$$

The corresponding quadratic equation with solutions $a \pm \sqrt{b}$ is $x^2 = 2ax + (b-a^2)$. Multiplying by x^{n-2} on both sides of the equation gives

$$x^n = 2ax^{n-1} + (b-a^2)x^{n-2}\,.$$

Since $a+\sqrt{b}$ and $a-\sqrt{b}$ are solutions to this equation,

$$\begin{aligned}(a+\sqrt{b})^n &= 2a(a+\sqrt{b})^{n-1} + (b-a^2)(a+\sqrt{b})^{n-2}\\ (a-\sqrt{b})^n &= 2a(a-\sqrt{b})^{n-1} + (b-a^2)(a-\sqrt{b})^{n-2}\end{aligned}$$

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By adding these two equations, we get:

$$(a+\sqrt{b})^n+(a-\sqrt{b})^n = 2a\big((a+\sqrt{b})^{n-1}+(a-\sqrt{b})^{n-1}\big)+(b-a^2)\big((a+\sqrt{b})^{n-2}+(a-\sqrt{b})^{n-2}\big)\,.$$

We recognise the terms $s_k = (a+\sqrt{b})^k+(a-\sqrt{b})^k$ for $k = n-2, n-1, n$ and can re-write the equation as follows:

$$s_n = 2a\,s_{n-1} + (b-a^2)s_{n-2}\,.$$

Note also that

$$\begin{aligned} s_1 &= (a+\sqrt{b}) \;\; + (a-\sqrt{b}) \;\; = 2a \\ s_2 &= (a+\sqrt{b})^2 + (a-\sqrt{b})^2 = 2a^2+2b\,. \end{aligned}$$

Hence, we get that the sequence s_n is defined by the recursive relation

$$s_n = 2a\,s_{n-1} + (b-a^2)s_{n-2}$$

with initial conditions $s_1 = 2a$ and $s_2 = 2(a^2+b)$.

Returning to the original sequence $c_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$, we see that $a = 1$ and $b = 2$ and that the sequence c_n is therefore determined by the recursive relation

$$c_n = 2\,c_{n-1} + c_{n-2}$$

with initial conditions $s_1 = 2$ and $s_2 = 6$.

3 Pell's Equation

Pell's Equation is the Diophantine (integer) equation of the form

$$x^2 - dy^2 = 1$$

where d is any natural, non-square number. Solutions (x, y) to Pell's Equations must be integers satisfying the equation above.

To solve Pell's Equation, we can first find an initial, or fundamental, solution by trial and error. First, we rearrange the equation to be $x^2 = 1 + dy^2$. By trial and error, we can try different values of y to find a value of $1 + dy^2$ that is a square. For any value y for which $1 + dy^2$ is a square, we get $x = \sqrt{1+dy^2}$. This gives the solution (x, y).

Suppose that (a, b) is a fundamental solution to Pell's Equation, and let $\alpha = a+b\sqrt{d}$. Let $\mathbb{Z}[\sqrt{d}]$ be the set of numbers of the form $x + y\sqrt{d}$ where $x, y \in \mathbb{Z}$. These include $\alpha = a+b\sqrt{d}$. For any number $u = x+y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, define $N(u) = x^2-y^2d$ and note that $N(\alpha) = a^2 - b^2d$. Since (a, b) satisfies Pell's equation, we see that $N(\alpha) = a^2 - b^2d = 1$.

For any two numbers $u = x + y\sqrt{d}$ and $v = p + q\sqrt{d}$ in $\mathbb{Z}[\sqrt{d}]$,

$$uv = (px + qyd) + \sqrt{d}(py + qx)\,.$$

We see that uv is also a number in $\mathbb{Z}[\sqrt{d}]$. Furthermore,

$$N(uv) = (px + qyd)^2 - (py + qx)^2d = (p^2 - q^2d)(x^2 - y^2d) = N(u)N(v)\,.$$

We can use this property to see that $N(u^2) = N(u)^2$. Generalizing this, we see that $N(u^n) = N(u)^n$ for all positive integers n.

Applying these observation to α, we see that α^n is a number in $\mathbb{Z}[\sqrt{d}]$ and can therefore by written as $\alpha^n = a_n + b_n\sqrt{d}$ for some integers a_n and b_n. Furthermore, $N(\alpha^n) = N(\alpha)^n = 1$ since $N(\alpha) = 1$, so the coefficient of a_n and b_n satisfy the equation $a_n^2 - b_n^2 d = 1$. In other words, the coefficients α^n are also solutions to Pell's Equation.

From this, we can conclude that if (a, b) satisfies Pell's Equation and $\alpha = a + b\sqrt{d}$, then the coefficients a_n and b_n of $\alpha^n = a_n + b_n\sqrt{d}$ also form a solution (a_n, b_n) to Pell's Equation.

Recursive Solutions to Pell's Equation

Since α^n yields coefficients which are solutions to Pell's equation, and $\alpha = a + b\sqrt{d}$ (where (a, b) are fundamental solutions), we can find a recursive relation for the solutions as described above.

First, let us form a quadratic in the form $x^2 + px + q$ with solutions $a \pm b\sqrt{d}$.

$$\text{Sum of roots:} \quad (a + b\sqrt{d}) + (a - b\sqrt{d}) = 2a = -p$$
$$\text{Product of roots:} \quad (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - b^2 d = q$$

With the sum and product of roots, we get the quadratic equation $\alpha^2 - 2a\alpha + a^2 - b^2 d = 0$, which has solutions $a \pm b\sqrt{d}$.

Rewriting the equation and multiplying both sides by α^{n-2}, we get

$$\alpha^n = 2a\alpha^{n-1} + (b^2 d - a^2)\alpha^{n-2}$$

This gives us the recursion for α^n. We can use this to get the recursion for the *n*th solution (x_n, y_n) to Pell's equation through simply substituting x or y in the place of α in this recursion. Doing this, we get

$$x_n = 2ax_{n-1} + (b^2 d - a^2)x_{n-2}\,,$$

with the initial conditions of $x_1 = a$, $x_2 = a^2 + b^2 d$ and $y_n = 2ay_{n-1} + (b^2 d - a^2)y_{n-2}$ with initial conditions $y_1 = b$, $y_1 = 2ab$.

General Formula for nth Solution

The general formula of x_n and y_n through looking at the expansions for smaller values. Let's first focus on the general formula for x_n:

We can use the recursive formula to look at the first 7 values of x_n:

$$\begin{aligned}
x_1 &= a \\
x_2 &= a^2 + b^2 d \\
x_3 &= a^3 + 3ab^2 d \\
x_4 &= a^4 + 6a^2 b^2 d + b^4 d^2 \\
x_5 &= a^5 + 10a^3 b^2 d + 5ab^4 d^2 \\
x_6 &= a^6 + 15a^4 b^2 d + 15a^2 b^4 d^2 + b^6 d^3 \\
x_7 &= a^7 + 21a^5 b^2 d + 35a^3 b^4 d^2 + 7ab^6 d^3 \,.
\end{aligned}$$

While it may be difficult to identify any pattern here, we can write out the coefficients of each term in terms of their binomial coefficients.

$$\begin{aligned}
x_1 &= \binom{1}{0} a^1 (b^2 d)^0 \\
x_2 &= \binom{2}{0} a^2 (b^2 d)^0 + \binom{2}{2} a^0 (b^2 d)^1 \\
x_3 &= \binom{3}{0} a^3 (b^2 d)^0 + \binom{3}{2} a^1 (b^2 d)^1 \\
x_4 &= \binom{4}{0} a^4 (b^2 d)^0 + \binom{4}{2} a^2 (b^2 d)^1 + \binom{4}{4} a^0 (b^2 d)^2 \\
x_5 &= \binom{5}{0} a^5 (b^2 d)^0 + \binom{5}{2} a^3 (b^2 d)^1 + \binom{5}{4} a^1 (b^2 d)^2 \\
x_6 &= \binom{6}{0} a^6 (b^2 d)^0 + \binom{6}{2} a^4 (b^2 d)^1 + \binom{6}{4} a^2 (b^2 d)^2 + \binom{6}{6} a^0 (b^2 d)^3 \\
x_7 &= \binom{7}{0} a^7 (b^2 d)^0 + \binom{7}{2} a^5 (b^2 d)^1 + \binom{7}{4} a^3 (b^2 d)^2 + \binom{7}{6} a^1 (b^2 d)^3 \,.
\end{aligned}$$

From this, a pattern becomes quite clear:

$$x_n = \binom{n}{0} a^n (b^2 d)^0 + \binom{n}{2} a^{n-2} (b^2 d)^2 + \binom{n}{4} a^{n-4} (b^2 d)^3 + \binom{n}{6} a^{n-6} (b^2 d)^4 + \cdots \,.$$

We can write this more concisely as:

$$x_n = \sum_{k=0}^{n} \binom{n}{2k} a^{n-2k} (b^2 d)^k \,.$$

A similar methodology can be followed to find y_n. First, we can look at the first few values of y_n:

$$
\begin{aligned}
y_1 &= \binom{1}{1} a^0 (b^2d)^0 b \\
y_2 &= \binom{2}{1} a^1 (b^2d)^0 b \\
y_3 &= \binom{3}{1} a^2 (b^2d)^0 b + \binom{3}{3} a^0 (b^2d)^1 b \\
y_4 &= \binom{4}{1} a^3 (b^2d)^0 b + \binom{4}{3} a^1 (b^2d)^1 b \\
y_5 &= \binom{5}{1} a^4 (b^2d)^0 b + \binom{5}{3} a^2 (b^2d)^1 b + \binom{5}{5} a^0 (b^2d)^2 b \\
y_6 &= \binom{6}{1} a^5 (b^2d)^0 b + \binom{6}{3} a^3 (b^2d)^1 b + \binom{6}{5} a^1 (b^2d)^2 b \\
y_7 &= \binom{7}{1} a^6 (b^2d)^0 b + \binom{7}{3} a^4 (b^2d)^1 b + \binom{7}{5} a^2 (b^2d)^2 b + \binom{7}{7} a^0 (b^2d)^3 b \,.
\end{aligned}
$$

From this, a pattern becomes quite clear.

$$
y_n = b \left(\binom{n}{1} a^{n-1} (b^2d)^0 + \binom{n}{3} a^{n-3} (b^2d)^1 + \binom{n}{5} a^{n-5} (b^2d)^3 + \binom{n}{7} a^{n-7} (b^2d)^4 + \cdots \right) \,.
$$

We can write this more concisely as

$$
y_n = b \sum_{k=0}^{n} \binom{n}{2k+1} a^{n-(2k+1)} (b^2d)^k \,.
$$

To summarize, for Pell's equation in the form $x^2 - dy^2 = 1$ with the fundamental solutions (a, b), with the *n*th solution (x_n, y_n), x_n can be found as

$$
x_n = \sum_{k=0}^{n} \binom{n}{2k} a^{n-2k} (b^2d)^k
$$

and y_n can be found as

$$
y_n = b \sum_{k=0}^{n} \binom{n}{2k+1} a^{n-(2k+1)} (b^2d)^k \,.
$$

References

[1] R. Heyman, Strange irrational powers, *Parabola* **56 (3)** (2020), 5 pages, https://www.parabola.unsw.edu.au/2020-2029/volume-56-2020/issue-3/article/strange-irrational-powers, last accessed on 2021-08-20.

[2] X. Hu, Discovering Companion Pell Numbers, *Parabola* **57 (1)** (2021), 5 pages, https://www.parabola.unsw.edu.au/2020-2029/volume-57-2021/issue-1/article/discovering-companion-pell-numbers, last accessed on 2021-08-20.