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Solutions 1641–1650

Q1641 Let a, b, c, d be positive real numbers such that $a+b+c+d=4$; prove that

$$\left(\frac{16}{a^2}-1\right)\left(\frac{16}{b^2}-1\right)\left(\frac{16}{c^2}-1\right)\left(\frac{16}{d^2}-1\right) \geq 15^4 .$$

SOLUTION We use the AGM (arithmetic–geometric–mean) inequality, which states that the average of any n positive real numbers is greater than or equal to the nth root of their product. Taking firstly the five numbers a, b, c, d, a and then the three numbers b, c, d, this gives

$$\frac{a+b+c+d+a}{5} \geq \sqrt[5]{abcda} \quad \text{and} \quad \frac{b+c+d}{3} \geq \sqrt[3]{bcd} .$$

Since $a+b+c+d=4$, these inequalities can be written

$$4+a \geq 5\sqrt[5]{a^2bcd} \quad \text{and} \quad 4-a \geq 3\sqrt[3]{bcd}$$

and then multiplied to give

$$16-a^2 \geq 15\,\sqrt[5]{a^2bcd}\,\sqrt[3]{bcd} .$$

By similar arguments, we obtain inequalities for $16-b^2$ and $16-c^2$ and $16-d^2$; multiplying them and collecting surds on the right hand side gives

$$\begin{aligned}&(16-a^2)(16-b^2)(16-c^2)(16-d^2)\\ &\qquad\qquad \geq 15^4\,\sqrt[5]{a^2bcd\,ab^2cd\,abc^2d\,abcd^2}\,\sqrt[3]{bcd\,acd\,abd\,abc} ,\end{aligned}$$

which simplifies to

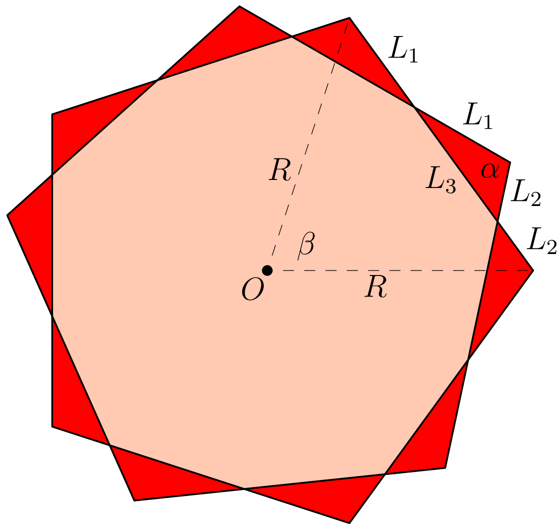
$$(16-a^2)(16-b^2)(16-c^2)(16-d^2) \geq 15^4a^2b^2c^2d^2 .$$

And now dividing both sides by the positive number $a^2b^2c^2d^2$ solves the problem.

Q1642 A regular n–gon is rotated by some angle about its centre O and the result is superimposed upon the original; the diagram illustrates the situation for $n=5$. Let A_0 be the area of the original polygon and P_0 its perimeter. Let A be the area common to both polygons (light red in the figure) and P the perimeter of the combined polygons (the whole coloured region in the figure). Prove that

$$\frac{P}{P_0}+\frac{A}{A_0}=2 .$$

SOLUTION By symmetry, all the dark red triangles in the diagram are congruent. (If desired, more details are given at the end of this solution.) Therefore we can label lengths as shown in the diagram.



Let S_0 be the length of one side of the original polygon; let R be the distance from the centre to one vertex (that is, the circumradius of the polygon); let α be the internal angle at a vertex; and let β be the angle subtended at the centre by a side. Then by standard geometrical arguments we have

$$\alpha = \pi - \beta \, , \quad P_0 = nS_0 \, , \quad S_0 = 2R\sin(\beta/2) \, , \quad A_0 = \tfrac{1}{2}nR^2\sin\beta \, .$$

The boundary of the combined figure consists of the sides L_1, L_2 taken $2n$ times, and the area of overlap is the area of the original, less n of the dark red triangles. If we write T for the area of one of these triangles, then

$$P = 2n(L_1 + L_2) \, , \quad A = A_0 - nT \, .$$

Since the triangle has sides L_1, L_2 with included angle α, we have

$$T = \tfrac{1}{2}L_1L_2\sin\alpha = \tfrac{1}{2}L_2L_2\sin\beta \, .$$

Applying the cosine rule to this triangle and noting that $L_3 = S_0 - L_1 - L_2$ yields

$$(S_0 - L_1 - L_2)^2 = L_1^2 + L_2^2 - 2L_1L_2\cos\alpha \, .$$

If we expand and rearrange this, using the identity

$$1 + \cos\alpha = 1 - \cos\beta = 2\sin^2(\beta/2) \, ,$$

we obtain (check it for yourself!)

$$2S_0(L_1 + L_2) = S_0^2 + 4L_1L_2\sin^2(\beta/2) \, .$$

Hence

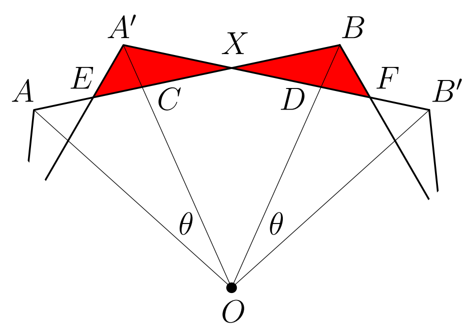
$$\begin{aligned}\frac{P}{P_0} &= \frac{2n(L_1+L_2)}{nS_0}\\ &= \frac{S_0^2+4L_1L_2\sin^2(\beta/2)}{S_0^2}\\ &= 1+\frac{4L_1L_2\sin^2(\beta/2)}{4R^2\sin^2(\beta/2)}\frac{n\sin\beta}{n\sin\beta}\\ &= 1+\frac{8nT}{8A_0}\end{aligned}$$

and finally

$$\frac{P}{P_0}+\frac{A}{A_0}=1+\frac{nT}{A_0}+\frac{A_0-nT}{A_0}=2$$

as claimed.

To confirm that all the dark red triangles are congruent, consider the following diagram, in which θ denotes the angle by which the original polygon is rotated.



We have

- $OA = OB' = R$ and $\angle AOC = \angle B'OD$ and $\angle OAC = \angle OB'D = \frac{1}{2}\alpha$, so $\triangle OAC \equiv \triangle OB'D$;
- therefore $OC = OD$; and $OA' = OB = R$, so by subtraction $A'C = BD$;
- also $\angle ACO = \angle B'DO$, therefore $\angle A'CX = \angle BDX$;
- and $\angle CA'X = \angle DBX = \frac{1}{2}\alpha$, so $\triangle CA'X \equiv \triangle DBX$;
- so $A'X = BX$ and $\angle A'XE = \angle BXF$ and $\angle XA'E = \angle XBF = \alpha$;

and so $\triangle A'XE \equiv \triangle BXF$, which is what we wanted to prove.

Q1643 A positive integer with k digits $d_0d_1\cdots d_{k-1}$ in base 10 is called a Geezer number if the digits consist of exactly d_0 zeros, exactly d_1 ones, exactly d_2 twos and so on. The number of digits is at most 10. For example, 2020 is a Geezer number since the digits $0,1,2,3$ occur $2,0,2,0$ times. We do not allow the first digit of a positive integer to be zero. Prove that in a k–digit Geezer number

(a) the sum of the digits is k;

(b) the digits $d_3, d_4, \ldots, d_{k-1}$ cannot be greater than 1.

SOLUTION To prove (a) we simply note that the sum of the digits $d_0 + d_1 + \cdots$ is the number of 0s in the Geezer number, plus the number of 1s, and so on, which is the total number of digits, which is k. For (b), let n be a k–digit Geezer number, let $i \geq 3$ and suppose that the digit i occurs j times, where $j \geq 2$. Then there are j digits which occur at least 3 times each. If $j = 2$ then these digits are not i (because i occurs twice, not three times or more); if $j \geq 3$ then one of the j digits may be i, but at least two are not. So n contains, as a minimum, the digits

$$a,\ a,\ a,\ b,\ b,\ b,\ i,\ i,$$

where a, b, i are all different. There may be up to two further digits. The sum of these digits is

$$S = 3(a + b) + 2i\,.$$

We have

$$S \geq 3(0 + 1) + 2(3) = 9\,,$$

and we know that $S \leq 10$, so there cannot be another digit i; so i occurs exactly twice, and one of the digits in n must be a 2. If either a or b is 2 then

$$S \geq 3(0 + 2) + 2(3) = 12$$

which is impossible; if not, then 2 is an extra digit and we have

$$S \geq 3(0 + 1) + 2 + 2(3) = 11$$

which is still impossible. We have ruled out all options; therefore it is impossible for a digit $i \geq 3$ to occur two or more times in a Geezer number.

Q1644 Of the students in a senior maths class, the proportion who read *Parabola* is 66%, *to the nearest percent*. What is the smallest possible number of students in the class?

SOLUTION Suppose that there are y students in the class, and x of them read *Parabola*. Then

$$65\frac{1}{2}\,\% < \frac{x}{y} < 66\frac{1}{2}\,\%\,,$$

which can be written

$$-\frac{1}{2}\% < \frac{x}{y} - \frac{66}{100} < \frac{1}{2}\%\,,$$

or

$$\left|\frac{x}{y} - \frac{33}{50}\right| < \frac{1}{200}\,. \tag{1}$$

We want to find positive integers x, y satisfying this inequality, with the smallest possible value of y; since $x = 33$, $y = 50$ is an obvious solution, we can limit ourselves to solutions with $y \leq 50$. The inequality can be rewritten

$$|50x - 33y| < \frac{y}{4}\,,$$

and we note that the left hand side is an integer.

For information about solving $50x - 33y = c$, where c is a given integer and x, y are required to be integers, see `www.parabola.unsw.edu.au/ 2010-2019/volume-49-2013/issue- -equations`. By an easy trial and (no) error, the equation $50x - 33y = 1$ has a solution $x = 2$, $y = 3$; so the general solution of

$$50x - 33y = c$$

is

$$x = 2c + 33t\,, \quad y = 3c + 50t\,,$$

where t is an integer. Since we are looking for $y \le 50$, we shall choose the value of t to guarantee this.

First note that $|c| < y/4 \le 12\frac{1}{2}$, so $c = 0, \pm 1, \pm 2, \ldots, \pm 12$, and that $c = 0$ gives the solution $x = 33$, $y = 50$ which we know already. If $c > 0$ then $3 \le 3c \le 36$; this is in the range 1 to 50 already, so we take $t = 0$ and $y = 3c$. But then we have $c < 3c/4$, which is impossible; so this case is ruled out. Therefore we have $c < 0$; write $c = -d$. Then $-36 \le 3c \le -3$, so to obtain y in the range 1 to 50 we need $t = 1$ and

$$y = -3d + 50\,. \tag{2}$$

Hence

$$d = |c| < \frac{-3d + 50}{4}\,,$$

which simplifies to $d \le 7$. To obtain the smallest possible y in (2) we need the largest possible d; so

$$d = 7\,, \quad x = 19\,, \quad y = 29\,,$$

and the smallest possible number of students in the class is 29.

Check: $\frac{19}{29} = 0.6551\cdots$, which is between $65\frac{1}{2}\%$ and $66\frac{1}{2}\%$.

Q1645 A row of boxes contains m zeros, followed by the numbers $1, 2, \ldots, n$ once each. There is a row of empty boxes below it. We give an example with $m = 3$ and $n = 5$.

0	0	0	1	2	3	4	5

We want to write the same numbers in the second row in such a way that no column contains the same number twice. Determine the number of ways of doing this (a) if $n = m$; (b) if $n = m + 1$; (c) if $n = m + 2$.

SOLUTION

(a) If $n = m$ then the m zeros in the second row must occupy the rightmost m spaces, and the other numbers can be arranged in any way in the leftmost m spaces. There are $m!$ ways of doing this.

(b) If $n = m + 1$ then all but one of the rightmost $m + 1$ places must be occupied by zeros. There are $m+1$ ways to choose this place; it can then be filled by any of the numbers $1, 2, \ldots, m + 1$ except that in the same column, so there are m options. The numbers not yet used are not zero, so they can go below the zeros in any way: $m!$ options. The total number of ways to fill in the numbers is

$$(m + 1)m\,m! = m(m + 1)!\,.$$

(c) If $n = m + 2$ then the zeros in the second row must occupy all but two of the rightmost $m+2$ spaces; the number of ways to choose these spaces is $C(m+2, 2) = \frac{1}{2}(m + 2)(m + 1)$. Suppose the unused spaces are in columns a and b. If a in the second row is written in column b, then the remaining columns contains 0s and a; these numbers do not occur among the $m + 1$ numbers yet to be placed, so there are $(m + 1)!$ possible placements. If a is not written in column b then it must be placed in one of the 0 columns (m options); then b must be placed in another of the 0 columns or in the a column (m options); and the remaining m numbers do not include 0 or a or b, so they can be placed in any way ($m!$ options). So the total number of arrangements is

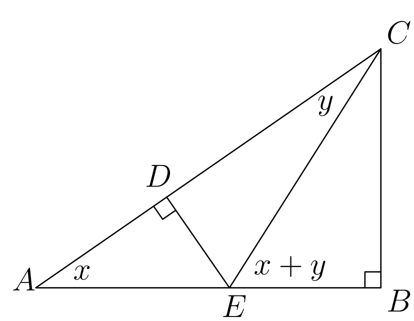
$$\begin{aligned} &C(m + 2, 2)[(m + 1)! + m\,m\,m!] \\ &\qquad = \frac{1}{2}(m + 2)(m + 1)(m^2 + m + 1)m! \\ &\qquad = \frac{1}{2}(m^2 + m + 1)(m + 2)!\,. \end{aligned}$$

Q1646 Triangle ABC has a right angle at B, and D is a point on the hypotenuse AC. The perpendicular to AC at D intersects AB at E, and we draw the line EC.

Use this diagram to prove the "cosine of a sum" formula

$$\cos(x + y) = \cos x \cos y - \sin x \sin y\,.$$

SOLUTION Let $\angle DAE = x$ and $\angle DCE = y$. Then $\angle CEB = x + y$ (exterior angle of a triangle equals the sum of the two opposite interior angles).



Moreover, Pythagoras' Theorem in $\triangle ADE$ gives $AD^2 + DE^2 = AE^2$; also $\triangle ADE$ and $\triangle ABC$ are similar (two equal angles), so $AD/AE = AB/AC$, so $AD \cdot AC = AB \cdot AE$.

Using these facts we have

$$
\begin{aligned}
\cos x \cos y - \sin x \sin y &= \frac{AD}{AE}\frac{CD}{CE} - \frac{DE}{AE}\frac{DE}{CE} \\
&= \frac{AD(AC - AD) - DE^2}{AE \cdot CE} \\
&= \frac{AD \cdot AC - AE^2}{AE \cdot CE} \\
&= \frac{AB \cdot AE - AE^2}{AE \cdot CE} \\
&= \frac{AB - AE}{CE} = \frac{BE}{CE} = \cos(x+y)
\end{aligned}
$$

as claimed.

Q1647 A monk visits t temples and burns a number of incense sticks, the same number at each temple. The temples are located on different islands in a magic lake and he visits them by boat. The lake doubles the number of sticks he holds each time he reaches an island. At the end of the day he has burnt all his incense sticks. How many, at least, did he start with?

SOLUTION Suppose that the monk starts with n sticks and burns s at each temple. Let $f(k)$ be the number of sticks he holds after leaving the kth temple. Then we have

$$f(t) = 2f(t-1) - s\,, \quad f(t-1) = 2f(t-2) - s$$

and so on. We write out these equations and multiply the second by 2, the third by 4, the fourth by 8 and so on. This gives

$$
\begin{aligned}
f(t) &= 2f(t-1) - s \\
2f(t-1) &= 4f(t-2) - 2s \\
4f(t-2) &= 8f(t-3) - 4s \\
&\;\;\vdots \\
2^{t-2}f(2) &= 2^{t-1}f(1) - 2^{t-2}s \\
2^{t-1}f(1) &= 2^t f(0) - 2^{t-1}s\,,
\end{aligned}
$$

where $f(0)$ is the number he had before visiting the first temple. Adding up all these equations, nearly all the f terms cancel and we obtain

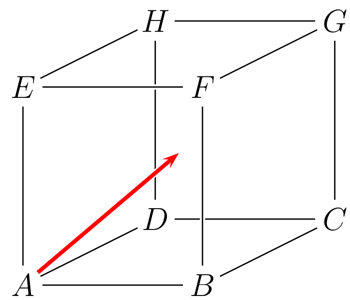
$$f(t) = 2^t f(0) - s(1 + 2 + 4 + \cdots + 2^{t-2} + 2^{t-1})\,.$$

The term in brackets is a geometric progression; adding the progression and remembering that $f(0) = n$ gives

$$f(t) = 2^t n - (2^t - 1)s\,.$$

But since the monk ended up with no sticks we have $f(t) = 0$ and hence $2^t n = (2^t - 1)s$. As $2^t - 1$ and 2^t have no common factor, n must be a multiple of $2^t - 1$. Therefore the smallest possible number of incense sticks the monk began with is $2^t - 1$.

Q1648 A rectangular box $ABCDEFGH$ has side lengths $AB = 19$, $AD = 20$, $AE = 21$, and has a small aperture at each of the vertices. A particle P is projected from A to the interior of the box along the line $x = y = z$. For instance, if the origin is at $A = (0, 0, 0)$ and the x, y, z axes are along AB, AD, AE respectively, the particle P will first rebound from the wall $BCGF$ at the coordinate $(19, 19, 19)$. Which vertex will P emerge from eventually?



SOLUTION When the particle hits a vertex, it must have travelled all the way across the box in the AB direction a number of times. Therefore the distance x travelled in this direction must be a multiple of 19. For similar reasons, the distance y travelled in the AD direction must be a multiple of 20 and the distance z in the AE direction is a multiple of 21. But $x = y = z$, so each of these distances must be a multiple of 19 and of 20 and of 21, and therefore a multiple of their product $19 \times 20 \times 21$. This means that when the particle falls into a vertex, it has travelled back and forth in the AB direction 20×21 times; this is an even number, so the particle has hit the face $ADHE$. Similarly, it has traversed the box in the AD direction 19×21 times, an odd number, and has ended up hitting the face $DCGH$; and in the AE direction 19×20 times, hitting $ABCD$. Therefore the vertex that the particle falls into must be one which is common to all three of these faces, and the only such vertex is D.

Q1649

(a) Given positive real numbers p, q, find all real numbers x, y with $x > y \geq 0$ such that
$$\sqrt{x} - \sqrt{y} = p \quad \text{and} \quad x - \sqrt{xy} + y = q\,.$$

(b) Given positive real numbers s, t, consider the simultaneous equations
$$x^2 + y^2 = s \quad \text{and} \quad 2x\sqrt{xy} - 3xy + 2y\sqrt{xy} = t\,.$$
Show that the equations have no solution if $s < 2t$; and find all solutions $x > y \geq 0$ if $s \geq 2t$.

SOLUTION In part (a), we can write the second equation as
$$x - (\sqrt{x} - \sqrt{y})\sqrt{y} = q\,;$$

using the first equation, this becomes

$$\begin{aligned} x - q = p(\sqrt{x} - p) \quad &\Rightarrow \quad x - (q - p^2) = p\sqrt{x} \\ &\Rightarrow \quad (x - (q - p^2))^2 = p^2 x \,. \end{aligned}$$

A similar process gives

$$y - (q - p^2) = -p\sqrt{y} \quad \text{and so} \quad (y - (q - p^2))^2 = p^2 y \,.$$

Thus x and y are both solutions of the quadratic equation

$$(z - (q - p^2))^2 = p^2 z \,,$$

which simplifies to

$$z^2 - (2q - p^2)z + (q - p^2)^2 = 0 \,.$$

This quadratic has discriminant $(2q - p^2)^2 - 4(q - p^2)^2 = p^2(4q - 3p^2)$, and so there are two real solutions if and only if $4q > 3p^2$; in this case we also have $2q - p^2 > 0$, which means that the solutions are both positive. Since x is the larger of the two solutions we have

$$x = \frac{2q - p^2 + p\sqrt{4q - 3p^2}}{2} \quad \text{and} \quad y = \frac{2q - p^2 - p\sqrt{4q - 3p^2}}{2} \,.$$

Given the equations in part (b), we expand by the binomial theorem to get

$$\begin{aligned} (\sqrt{x} - \sqrt{y})^4 &= x^2 - 4x\sqrt{xy} + 6xy - 4y\sqrt{xy} + y^2 \\ &= s - 2t \end{aligned} \tag{1}$$

and

$$\begin{aligned} (x - \sqrt{xy} + y)^2 &= x^2 + xy + y^2 - 2x\sqrt{xy} + 2xy - 2y\sqrt{xy} \\ &= s - t \,. \end{aligned}$$

If $s < 2t$ then (1) is impossible and there is no solution. If $s > 2t$ then we have

$$\sqrt{x} - \sqrt{y} = p \quad \text{and} \quad x - \sqrt{xy} + y = q \,,$$

where

$$p = \sqrt[4]{s - 2t} \,, \quad q = \sqrt{s - t}$$

(taking positive roots since $x > y \geq 0$), and we may use the result of (a). We have

$$4q - 3p^2 = 4\sqrt{s - t} - 3\sqrt{s - 2t} > 3\sqrt{s - t} - 3\sqrt{s - 2t} > 0 \,,$$

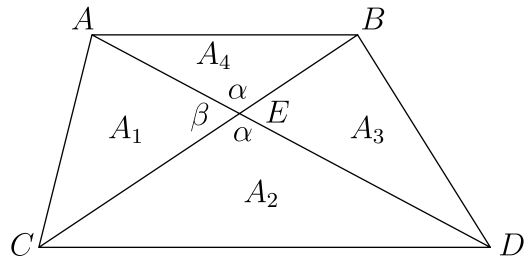
and so the solutions are

$$x = \frac{2\sqrt{s - t} - \sqrt{s - 2t} + \sqrt[4]{s - 2t}\sqrt{4\sqrt{s - t} - 3\sqrt{s - 2t}}}{2}$$

and

$$y = \frac{2\sqrt{s - t} - \sqrt{s - 2t} - \sqrt[4]{s - 2t}\sqrt{4\sqrt{s - t} - 3\sqrt{s - 2t}}}{2} \,.$$

Q1650 A trapezium $ABDC$ has AB parallel to CD. The diagonals AD and BC divide the trapezium into four triangles with areas A_1, A_2, A_3, A_4 as shown in the diagram.



(a) Prove that $A_1A_3 = A_2A_4$.

(b) Deduce that the area of the trapezium is at least $4A_1$. What more can you say about the trapezium if its area is $4A_1$?

SOLUTION With angles labelled as in the diagram, we have

$$\begin{aligned} A_1A_3 &= \left[\frac{1}{2}(EA)(EC)\sin\beta\right]\left[\frac{1}{2}(EB)(ED)\sin\beta\right] \\ &= \left[\frac{1}{2}(EC)(ED)\sin\alpha\right]\left[\frac{1}{2}(EA)(EB)\sin\alpha\right] \\ &= A_2A_4\ . \end{aligned}$$

For part (b), we begin by showing that $A_1 = A_3$. Since $AB \parallel CD$, the altitudes of $\triangle ACD$ and $\triangle BDC$ are equal; their bases are also equal; so their areas are equal. That is,

$$A_1 + A_2 = A_3 + A_2$$

and we have $A_1 = A_3$ as claimed. Now write $A_1 = A$, and let $A_2 = x$. From (a), the total area is

$$A_1 + A_2 + A_3 + A_4 = 2A + x + \frac{A^2}{x} = 4A + \left(\sqrt{x} - \frac{A}{\sqrt{x}}\right)^2 .$$

Since the square cannot be negative, the total area is at least $4A$. If the area is exactly $4A$ then the square is zero, which gives $x = A$, that is, $A_2 = A_1$. Therefore E bisects AD (triangles of equal area and equal altitude must have the same base); and from here we leave it up to you to show that $AB = CD$, and so the trapezium is a parallelogram.