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There's nothing square about squares!

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I thought I would share with you a few facts about squares - some well known, and others perhaps not so well known.

In our journey with numbers, squares are one of the first types of numbers we encounter in early schooling. The ancient Greeks were fascinated by them, and they had nice ways to visualise geometrically various identities involving squares.
For example, they proved that for any positive integer n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)\,,$$

a problem often given to senior students in high school to prove by induction.[2]

You will recall that square numbers must end in $0, 1, 4, 5, 6$ or 9, so, for example, the number 389808585038543 cannot be a square. There are other restrictions on what a square number can end in that you might like to think about.

Another interesting fact is that

$$1^2 + 2^2 + \cdots + 24^2 = 70^2\,,$$

and this is the only time the sum of the first n consecutive squares adds to a square (except of course for the trivial $1^2 = 1^2$).

The famous mathematician Euler proved the following remarkable fact in the 18th century:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \to \frac{\pi^2}{6}\,,$$

which means that the sum on the left gets as close as we please to $\frac{\pi^2}{6}$ the more terms we take. It is known (but slightly harder to prove) that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots \to \infty\,,$$

where the sum on the left is the sum of the reciprocals of the prime numbers. Thus, the sum continues to grow without bound the more terms you take. This tells us that in some sense there are *more* primes than squares!

By the way, if we change every second sign in the sum of reciprocals of the squares then we obtain half the sum, that is:

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \to \frac{1}{2} \times \frac{\pi^2}{6} = \frac{\pi^2}{12}\,.$$

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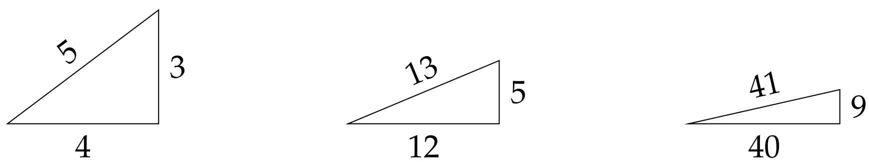
[2] Can you think of a picture proof of this identity? One such proof is given in the *Parabola* article [3].

Pythagorean Triples

Sets of positive integers which can be the side lengths of a right-angles triangle are called *Pythagorean Triples*. You will recall the well-known examples such as



(3,4,5) (5,12,13) (9,40,41)

Plato mentions one simple way to find some of these: if n is a positive integer, then $(n^2-1, 2n, n^2+1)$ is a Pythagorean triplet. For example if $n=3$, then we have $(8,6,10)$.

A more general way of finding most of the triples is to take any two positive integers p and q with $p>q$, and form the triple $(p^2-q^2, 2pq, p^2+q^2)$. For example, if $p=2$ and $q=1$, then we have $(3,4,5)$. The triple $(9,12,15)$ cannot be formed in this way (you might like to convince yourself of that) but we can obtain *all* triples by taking a triple of the form above and multiplying each number by any positive integer you like. Hence, the most general parametrization of the triples is $\big(k(p^2-q^2), 2kpq, k(p^2+q^2)\big)$.

A *primitive triple* is one where the numbers have no common factors except 1. We can find all primitive triples by taking two positive integers p and q which are *co-prime* (i.e., have no common factor except 1) and *not both odd* and using the formula above. For example, taking the coprime integers $p=4$ and $q=3$, we have $(7,24,25)$.[3]

Sums of two squares

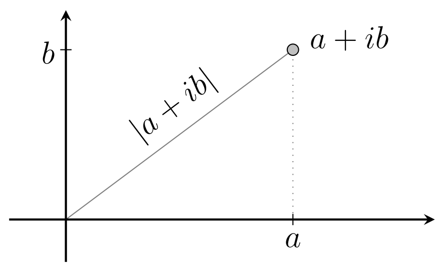
The number 13 can be expressed as the sum of two integer squares; e.g., $13 = 2^2+3^2$. On the other hand 19 cannot be so expressed (try it and see!). Both of these numbers are primes, so we will begin by thinking about what primes can be expressed as the sum of two squares. If you experiment with various primes and look for patterns, then you may observe that primes of the form $4k+1$ can be expressed as the sum of two squares and that primes of the form $4k-1$ cannot. (Note that 2 is prime and of neither shape but, of course, $2 = 1^2+1^2$). Note that $13 = 4\times 3+1$ whereas $19 = 4\times 5-1$. Here is the big result:

Theorem 1 (The Two-Square Theorem)**.** *If p is an odd prime, then p is the sum of two squares if and only if $p = 4k+1$ for some integer k.*

The proof of this result is not all that easy. Most number theory books will have one for you to consult. If you are familiar with complex numbers, then there is a very neat way to take two numbers, each of which is the sum of two squares, and express their product as the sum of two squares.

[3] You can find more information on Pythagorean triples in the *Parabola* articles [2, 4, 5, 7].

Recall that the square of the *modulus* of the complex number $a + ib$ (where $i^2 = -1$) is given by $|a + ib|^2 = a^2 + b^2$:



For example, we know that $13 = 2^2 + 3^2$ and $5 = 2^2 + 1^2$. We can write these numbers as $13 = |2 + 3i|^2$ and $5 = |2 + i|^2$. Now,

$$65 = 13 \times 5 = |2 + 3i|^2|2 + i|^2 = |(2 + 3i)(2 + i)|^2 = |1 + 8i|^2 = 1^2 + 8^2\,.$$

We could also write $13 = |3 + 2i|^2$ and $5 = |2 + i|^2$; then

$$65 = 13 \times 5 = |3 + 2i|^2|2 + i|^2 = |(3 + 2i)(2 + i)|^2 = |4 + 7i|^2 = 4^2 + 7^2\,.$$

This gives a nice way to write a number as the sum of two squares and also gives us a hint as to which numbers can be so written. We take any positive integer N and factorise it as follows:

$$N = M^2 p_1 p_2 \cdots p_m\,,$$

where M^2 is the largest square factor of N and $p_1, p_2, \ldots, p_m$ are primes. Then N can be expressed as the sum of the two squares if and only if each of the numbers $p_1, p_2, \ldots, p_m$ is of the form $4k + 1$. Moreover, if we can manually write each p_i as the sum of two squares, then we can use complex numbers to write N as the sum of two squares.

For example, if $N = 260$, then we write $N = 260 = 2^2 \times 13 \times 5$ and then, using the earlier calculation, we can write

$$N = 260 = 2^2(4^2 + 7^2) = 8^2 + 14^2\,.$$

Sums of three squares

As we saw, not all numbers are the sum of two squares. What about three squares? Well, it is easy to see that the number 7 is not the sum of three squares and neither is the number 23.

Euler stated that every number is the sum of three squares unless it is of the form $4^\ell(8k + 7)$ for some integers k and ℓ. He was not, however, able to prove it. That honour went to Gauss. It is not too hard to show that a number of this form is not the sum of three squares but the converse is much more difficult. Most number theory books skip it!

Sums of four squares

How many squares then, do we need in order to represent every number as a sum of squares? The answer happily is 4. Lagrange proved the following theorem.

Theorem 2. *Every positive integer is the sum of at most four integer squares.*

His proof is found in many number theory books. One can use an idea similar to one using complex numbers to show how to write the product of two numbers as the sum of four squares given that each number is expressed as the sum of four squares. This method uses *quaternions* rather than complex numbers. Alternatively, one can use the (equivalent) identity:

$$\begin{aligned}(a^2+b^2+c^2+d^2)(A^2+B^2+C^2+D^2) &= (aA+bB+cC+dD)^2 \\ &+ (aB-bA+cD-dC)^2 \\ &+ (aC-bD+cA-dB)^2 \\ &+ (aD+bC-cB-dA)^2\,.\end{aligned}$$

For example, we can express 7 and 23 each as the sum of four squares,

$$\begin{aligned}7 &= 2^2+1^2+1^2+1^2 \\ 23 &= 3^2+3^2+2^2+1^2\,.\end{aligned}$$

Hence, we can write their product as the sum of four squares:

$$\begin{aligned}161 = 7\times 23 = (2^2+1^2+1^2+1^2)(3^2+3^2+2^2+1^2) &= (2\times 3+1\times 3+1\times 2+1\times 1)^2 \\ &+ (2\times 3-1\times 3+1\times 1-1\times 2)^2 \\ &+ (2\times 2-1\times 1-1\times 3+1\times 3)^2 \\ &+ (2\times 1+1\times 2-1\times 3-1\times 3)^2 \\ &= 12^2+2^2+3^2+2^2\,.\end{aligned}$$

Sums of squares equal to a square

When setting test questions and exercises involving vectors, it is nice to be able to construct vectors with integer length. In three dimensions, the length of the vector (a, b, c) is $\sqrt{a^2+b^2+c^2}$. Thus, we seek integers a, b, c so that $a^2+b^2+c^2$ is a square.

There are various ways to do this. Here is a simple method. Take a prime p of the form $4k+1$. As we mentioned above, such a prime can be expressed as the sum of two squares, so

$$p = a^2+b^2 = 4k+1\,.$$

If we add $(2k)^2$ to both sides, then we obtain a square:

$$a^2+b^2+(2k)^2 = 4k^2+4k+1 = (2k+1)^2\,.$$

For example with $p = 17 = 4 \times 4 + 1$, we have $k = 4$ and we can write $17 = 4^2 + 1^2$. This gives

$$4^2 + 1^2 + 8^2 = 9^2$$

so the vector $(4, 1, 8)$ has length 9.

The prime $p = 37$ gives $k = 9$ and leads to $(1, 6, 18)$ which has length 19. By the way, as you may have guessed, the number p does not necessarily have to be prime; any number which is the sum of two squares will also work.

An even more demanding requirement is for the vector (a, b, c) to have both integer length and for the sum $a + b + c$ to also be a square!

I came up with the following simple algorithm, motivated by the above. Take any integer q and let $x = 2q^2$, $y = 2q$ and $z = 1$. Then, as before, $x^2 + y^2 + z^2 = (2q^2 + 1)^2$ but we also require $2q^2 + 2q + 1$ to be a square, say r^2. By multiplying by 2 and doing some simple algebra, we obtain

$$(2q + 1)^2 - 2r^2 = -1\,,$$

which you might recognise as a Pell-type equation. Using the basic theory of Pell equations, writing a_n for $2q+1$, we can obtain a sequence of values for q obtained from the recurrence

$$a_n = 6a_{n-1} - a_{n-2} \qquad \text{where} \qquad a_0 = 1\,,\;\; a_1 = 7\,.$$

Using the recurrence, we can make up a table of some values.

n	0	1	2	3
a_n	1	7	41	239
q	0	3	20	119

Using the formulae above, the value $q = 3$ gives the vector $(18, 6, 1)$ which has integer length and the sum of the entries is 25; the value $q = 20$ produces $(800, 40, 1)$, while $q = 119$ gives the vector $(28322, 238, 1)$ which again has integer length and the sum of the entries is 169^2.

These algorithms do not, of course, give all the solutions to the stated problems, just some of them. Here is another far less easy to motivate algorithm. I leave you to check that it does in fact work.

STEP 1 Choose integers p and q.
STEP 2 Factor the number $6(p + q)^2 + 2q^2$ into two even integers α and β.
STEP 3 Let $s = (\alpha + \beta)/2 + 3(p + q)$.
STEP 4 Then $X^2 + Y^2 + Z^2$ and $X + Y + Z$ are both squares, where

$$X = 2(p^2 + q^2 - s^2)\,,\; Y = 2\big((p - s)^2 - q^2 + p(q - s)\big) \quad \text{and} \quad Z = (q - s)^2 - p^2 + 4q(p - s)\,.$$

For example, with $p = 2$ and $q = -7$, we have $6(p + 1)^2 + 2q^2 = 248 = 4 \times 62$. So with $\alpha = 4$ and $\beta = 62$, we find $s = 18$. This produces $X = -542$, $Y = 314$ and $Z = 1069$ and

$$X + Y + Z = 29^2 \qquad \text{and} \qquad X^2 + Y^2 + Z^2 = 1239^2\,.$$

You might like to write a computer program to generate other examples.

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