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Two simple theorems and their applications

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1 Introduction

More than two years ago, while I was attending a physics class in high school, our teacher proposed the following problem: "Given a body of ssome fixed mass, how would you split it so that the gravitational force between the two parts is maximal?". If you know a little of physics, then you will immediately recognize that this problem can be rewritten as: "Which two numbers add to a given sum and have maximal product?". The answer is simple and elegant: split the mass into two equal parts! From this simple concept, I started a sort of mathematical journey. In the beginning, my goal was to generalize this question, trying to maximize the product of k numbers whose sum is fixed. I managed to solve this problem in a relatively short period of time. But everything changed when I found, through a simple google search, that this result is already well-known, even though it did not appear in any textbook I have ever read. In that moment, I decided to give this result much more attention, not just to proving it (something many people had already done on their own) but also to showing some of its many interesting applications. Among those, there is a "dual" theorem which involves a fixed product and trying to find the minimal possible sum for the factors.

2 The two theorems

Let $a_1, a_2, \ldots, a_n, L, V$ be positive real numbers. The two theorems are as follows.

Theorem 1. *If* $a_1 + a_2 + \cdots + a_n = L$, *then*

$$a_1 a_2 \cdots a_n \leq \left(\frac{L}{n}\right)^n .$$

Furthermore, there is equality if and only if $a_1 = a_2 = \cdots = a_n = \frac{L}{n}$.

Theorem 2. *If* $a_1 a_2 \cdots a_n = V$, *then*

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{V} .$$

Furthermore, there is equality if and only if $a_1 = a_2 = \cdots = a_n = \sqrt[n]{V}$.

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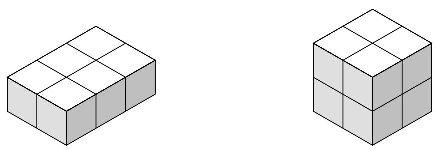
3 Proofs

Intuitively, the two theorems above seem obviously true. For $n = 2$, the rectangle with greatest area, given length and height sum $a_1 + a_2 = L$, is a square. For instance, the following rectangles both have length and height sum $1 + 3 = 2 + 2 = 4$ but the first has area $1 \times 3 = 3$ whereas the second is a square and has greater area: $2 \times 2 = 4$.

For $n = 3$, the box with greatest volume, given dimension sums $a_1 + a_2 + a_3 = L$, is a cube. For instance, the following boxes both have dimension sums $1 + 2 + 3 = 2 + 2 + 2 = 6$ but the first has volume $3 \times 2 \times 1 = 6$ whereas the second is a cube and has greater volume: $2 \times 2 \times 2 = 8$.



In general, the first theorem states that an n-dimensional box has greatest volume, given some dimension sum, when its dimensions are all equal. The second theorem states the same, only differently: an n-dimensional box has smallest dimension sum, given some volume, when its dimensions are all equal. If, instead of dimension sum, we measured perimeter, surfaces, and so on, then the objects of greatest volume would be spherical. For instance, the planar shape with greatest area inside a perimeter of given length is a circle, and the 3-dimensional shape with greatest volume inside a surface with given area is a sphere; we see this proved in real life by physical masses pulled into the shape of spheres by their own gravity.

Mathematically, we can prove the two theorems as follows.

Proof of Theorem 1. The theorem is certainly true for $n = 1$.

For $n = 2$, define $x = \frac{a_1+a_2}{2} - a_1 = \frac{L}{2} - a_1$; then $a_1 = L/2 - x$, $a_2 = L/2 + x$ and

$$a_1 a_2 = \left(\frac{L}{2} - x\right)\left(\frac{L}{2} + x\right) = \left(\frac{L}{2}\right)^2 - x^2 \leq \left(\frac{L}{2}\right)^2$$

with equality exactly when $x = 0$; that is, when $a_1 = a_2 = \frac{L}{2}$.

Suppose that $n \geq 3$ and that $a_1 a_2 \cdots a_n$ is greatest possible. Assume that $a_i \neq a_j$ for some i and j with $i < j$. Since the theorem is true for $n = 2$, it follows that $a_i a_j < a'_i a'_j$ where $a'_i = a'_j = (a_i + a_j)/2$. Then $a_1 + \cdots + a_i + \cdots + a_j + \cdots + a_n = L$ and

$$a_1 \cdots a_i \cdots a_j \cdots a_n < a_1 \cdots a'_i \cdots a'_j \cdots a_n$$

which contradicts the maximality of $a_1 a_2 \cdots a_n$. Therefore, the numbers $a_1, \ldots, a_n$ must be equal, and the theorem follows. □

Theorem 2 can be proved similarly. Instead, we give a different proof.

Proof of Theorem 2. Set $L = a_1 + a_2 + \cdots + a_n$. By Theorem 1,

$$V = a_1 a_2 \cdots a_n \leq \left(\frac{L}{n}\right)^n,$$

with equality exactly when $L = a_1 + a_2 + \cdots + a_n = \frac{L}{n}$. In other words,

$$a_1 + a_2 + \cdots + a_n = L \geq n\sqrt[n]{V}\ .$$

with equality exactly when $a_1 = a_2 = \cdots = a_n = L/n = \sqrt[n]{V}$. □

We see that Theorem 2 follows from Theorem 1, and it is easy to prove the converse; indeed, the two theorems are equivalent.

4 Applications

The two theorems are very simple and yet have some very interesting applications. This section shows some of these applications.

4.1 Arithmetic and geometric mean

The *arithmetic mean* and the *geometric mean* of positive numbers $a_1, \ldots, a_n$ are defined respectively as

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad \sqrt[n]{a_1 a_2 \cdots a_n}\,.$$

Theorem 1 implies the well-known fact that the geometric mean is less than or equal to the arithmetic mean:

$$\sqrt[n]{a_1 \cdots a_n} \leq \sqrt[n]{\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n} = \frac{a_1 + a_2 + \cdots + a_n}{n}\ .$$

4.2 Two nice inequalities

Let $a_k = \frac{1}{2^k}$ for $k = 1, \ldots, n$. Then

$$L = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} 2^{-k} = 1 - \frac{1}{2^n}$$

and

$$a_1 a_2 \cdots a_n = \prod_{k=1}^{n} 2^{-k} = 2^{-(1+2+\cdots+n)}\,.$$

We can now use the identity[2] $1 + 2 + \cdots + n = \binom{n+1}{2}$ where $\binom{n+1}{2}$ is the binomial coefficient equal to $\frac{n(n+1)}{2}$. By Theorem 1,

$$2^{-\binom{n+1}{2}} = a_1 a_2 \cdots a_n < \left(\frac{L}{n}\right)^n = \left(\frac{1 - \frac{1}{2^n}}{n}\right)^n < \frac{1}{n^n}\,.$$

This gives us the following nice inequality.

Lemma 3. *For each integer* $n \geq 1$, $\;2^{\binom{n+1}{2}} > n^n$.

Now define $a_k = \frac{1}{k+1}$ for $k = 1, \ldots, n$. Then

$$L = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} \frac{1}{k+1} \qquad \text{and} \qquad a_1 a_2 \cdots a_n = \prod_{k=1}^{n} \frac{1}{k+1} = \frac{1}{(n+1)!}\,.$$

Now, it is known that

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x} dx\,,$$

so, since $\frac{1}{x}$ is strictly decreasing for $x > 1$,

$$\ln(n+1) > \sum_{k=1}^{n} \frac{1}{k+1} = L\,.$$

Therefore, Theorem 1 implies that

$$\frac{1}{(n+1)!} = a_1 a_2 \cdots a_n < \left(\frac{L}{n}\right)^n < \left(\frac{\ln(n+1)}{n}\right)^n.$$

This gives us another nice inequality.

Lemma 4. *For each integer* $n \geq 1$, $\;n^n < (n+1)!\big(\ln(n+1)\big)^n$.

[2] For proofs of this identity, see the *Parabola* article Proof by picture: A selection of nice picture proofs.

4.3 The greatest maximal product

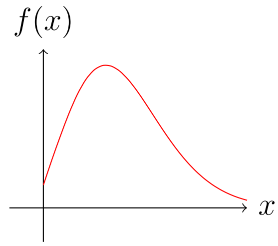
Consider a positive number L. We know from Theorem 1 that if positive numbers $a_1, a_2, \ldots, a_n$ have sum L, then their product has the upper bound

$$a_1 a_2 \cdots a_n \leq \left(\frac{L}{n}\right)^n .$$

We might ask: how large can this bound be? To answer this question, I define the function

$$f(x) = \left(\frac{L}{x}\right)^x .$$

As can be seen from a quick plot (or an easy proof), the function has a maximum:



To find this maximum, we solve $\frac{d}{dx} f(x) = 0$:

$$\begin{aligned} 0 = \frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{L}{x}\right)^x = \frac{d}{dx}\, e^{\ln((\frac{L}{x})^x)} = \frac{d}{dx}\, e^{x \ln \frac{L}{x}} &= e^{x \ln \frac{L}{x}} \frac{d}{dx}\, x \ln \frac{L}{x} \\ &= e^{x \ln \frac{L}{x}} \left(\ln \frac{L}{x} + x \frac{d}{dx} \ln \frac{L}{x} \right) \\ &= e^{x \ln \frac{L}{x}} \left(\ln \frac{L}{x} + x \frac{d}{dx} (\ln L - \ln x) \right) \\ &= e^{x \ln \frac{L}{x}} \left(\ln \frac{L}{x} + x \frac{1}{x} \right) \\ &= e^{x \ln \frac{L}{x}} \left(\ln \frac{L}{x} - 1 \right) . \end{aligned}$$

Since $e^{x \ln \frac{L}{x}} > 0$, we see that $\ln \frac{L}{x} - 1 = 0$; that is, $x = L/e$. We conclude that if we fix L and let n vary, then the maximal product $a_1 a_2 \cdots a_n$ of positive numbers with sum L is $n = L/e$ or, more precisely, one of the two integers closest to this number:

$$n = \left\lfloor \frac{L}{e} \right\rfloor \qquad \text{or} \qquad n = \left\lceil \frac{L}{e} \right\rceil .$$

The product then approximately equals

$$a_1 a_2 \cdots a_n = \left(\frac{L}{n}\right)^n = \left(\frac{L}{L/e}\right)^{L/e} = e^{\frac{L}{e}} .$$

Hence, I define the *greatest maximal product* of L to be this function:

$$GMP(L) = \max_x f(x) = e^{\frac{L}{e}} .$$

Two simple yet fundamental properties of this function are as follows.

Lemma 5. *(a) If $\ell < m$, then $GMP(\ell) < GMP(m)$; (b) $GMP(e \ln n) = n$.*

4.4 The minimal sum of factors of a number

Every natural number n has a unique factorization into a product of prime powers:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

We define

$$Z(n) = \sum_{i=1}^{k} p_i \alpha_i \,.$$

For example, $36 = 2^2 3^2$, $100 = 2^2 5^2$ and $51 = 3^1 17^1$, so

$$\begin{aligned} Z(36) &= 4 + 6 = 10 \\ Z(100) &= 4 + 10 = 14 \\ Z(51) &= 3 + 17 = 20\,. \end{aligned}$$

Then

$$\begin{aligned} n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} &= \overbrace{p_1 \cdots p_1}^{\alpha_1} \overbrace{p_2 \cdots p_2}^{\alpha_2} \cdots \overbrace{p_k \cdots p_k}^{\alpha_k} \\ &\leq GMP(\overbrace{p_1 + \cdots + p_1}^{\alpha_1} + \overbrace{p_2 + \cdots + p_2}^{\alpha_2} + \cdots + \overbrace{p_k + \cdots + p_k}^{\alpha_k}) \\ &= GMP(Z(n))\,. \end{aligned}$$

By Lemma 5 (b), $GMP(e \ln n) = n \leq GMP(Z(n))$. Lemma 5 (a) therefore implies the following result.

Lemma 6. $Z(n) \geq e \ln n$.

4.5 A surprising factorial inequality

Now let's consider the factorial $n!$, where n is a natural number. Since $\ln x$ is strictly increasing and positive for $x > 1$,

$$e \ln n! = e \sum_{i=1}^{n} \ln i \geq e \int_1^n \ln x \, dx = e\big[x \ln x - x\big]_1^n = e(n \ln n - n + 1)\,.$$

Now, by Lemma 5,

$$n! = GMP(e \ln n!) \geq GMP(e\, n \ln n - e\, n + e) = e^{\frac{e\, n \ln n - e\, n + e}{e}} = e^{n \ln n - n + 1}\,.$$

In other words, we have the following interesting inequality.

Lemma 7. *For each integer $n \geq 1$, $n! \geq \frac{e^{n \ln n + 1}}{e^n}$.*

5 Acknowledgements

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