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A spider cubic related to triangles

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1 Introduction

The gardener's approach for drawing an ellipse is well known: we fix two points and with a pen we pull a closed string around the two points tight; see Figure 1. Another way of looking at this procedure is the following: an ellipse is the set of points all of which are the third vertex of all triangles with given side and given perimeter.

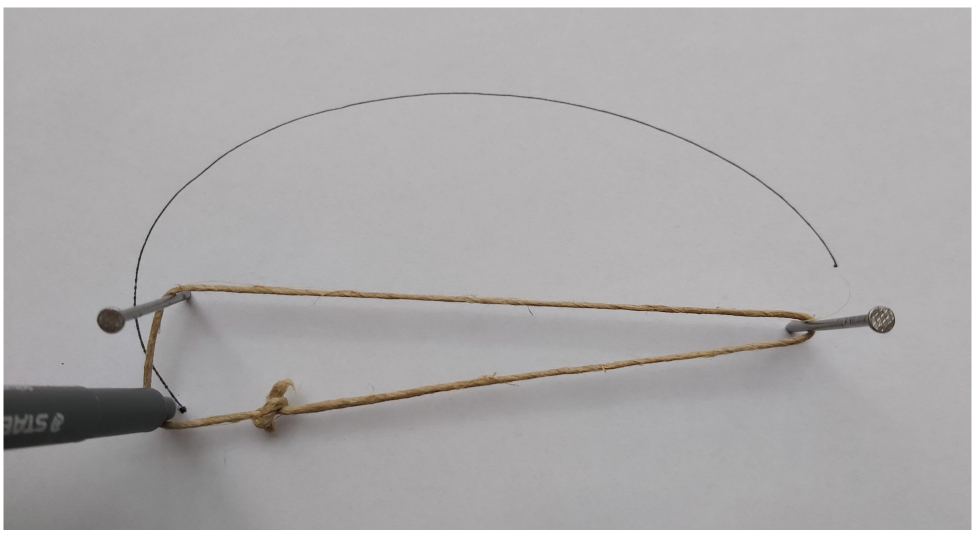

Figure 1: Two nails and a rope is all that is needed to draw an ellipse.

A natural question then is in which way this approach can be generalized to other invariants of the triangle or even to other figures. The other two most well known invariants of a triangle are the circumscribed and the inscribed circle (or incircle). Obviously, fixing a side and the circumscribed circle the collection of all third vertices of the corresponding triangles is precisely the circumscribed circle. In this note, we determine the curve that is the set of points P all of which are the third vertex of all triangles with a given side $[AB]$ and a given incircle $\omega(C, r)$ tangent anywhere to the line segment $[AB]$.

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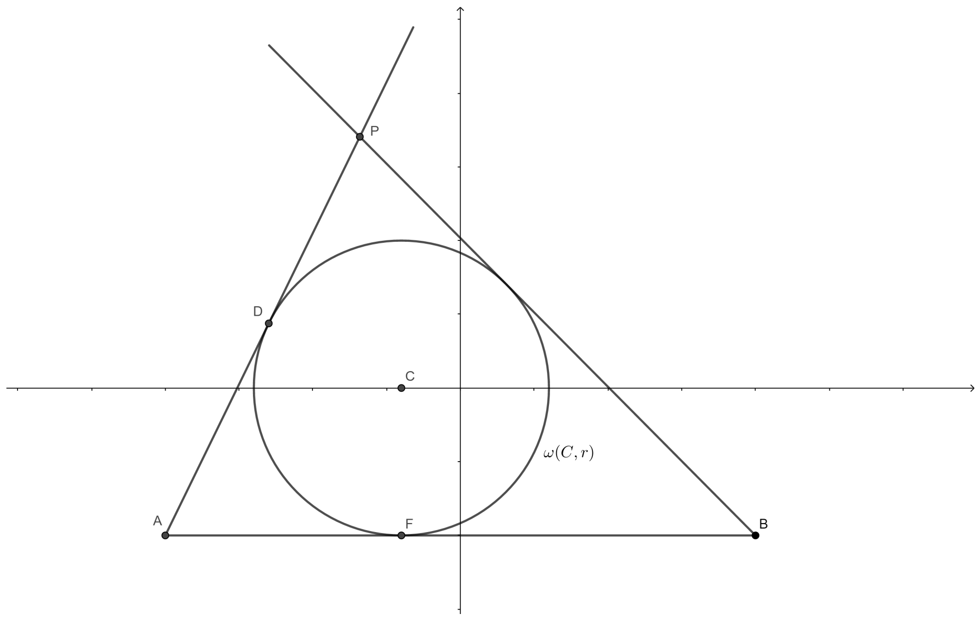

Figure 2: A triangle ΔABP with side $[AB]$ and incircle $\omega(C,r)$.

2 Finding the curve

We choose an orthonormal coordinate system with the y-axis being the perpendicular bisector of the line segment $[AB]$ and the x-axis parallel to $[AB]$ at a distance r above it. Then $A = (-a, -r)$ and $B = (a, -r)$ with $a \in \mathbb{R}_0^+$ and $r \in \mathbb{R}_0$, both fixed. Let $C = (c, 0)$, with $c \in \mathbb{R}$, be the center of the incircle $\omega(C, r)$ which is made to move along the x-axis.

We now construct the tangents to $\omega(C, r)$ from A and B as in Figure 2. Let F be the tangent point on $[AB]$ and $\omega(C, r)$. Because CF is perpendicular to $[AB]$, it follows that $\mid AF \mid = c + a$. If we denote by D the other tangent point from A to the circle $\omega(C, r)$, then $\mid AD \mid = \mid AF \mid = c + a$. Hence, $D = (x_1, y_1)$ lies on two circles, namely $\omega(C, r)$ and $\omega(A, a + c)$. Thus,

$$(x_1 - c)^2 + y_1^2 = r^2 \; ;$$
$$(x_1 + a)^2 + (y_1 + r)^2 = (a + c)^2 \; .$$

Solving this system for x_1 and y_1 gives

$$D = \left(c - \frac{2r^2(a + c)}{(a + c)^2 + r^2}, \frac{r\big((a + c)^2 - r^2\big)}{(a + c)^2 + r^2} \right) \; .$$

Since the tangent to $\omega(C, r)$ from A in D is perpendicular to the line CD, its slope m_{T_A} is related to the slope m_{CD} of the line CD by the following expression: $m_{T_A} \cdot m_{CD} = -1$. Thus, the tangent T_A has the equation

$$T_A : \; y + r = \frac{2r(a + c)}{(a + c)^2 - r^2}(x + a) \; . \tag{1}$$

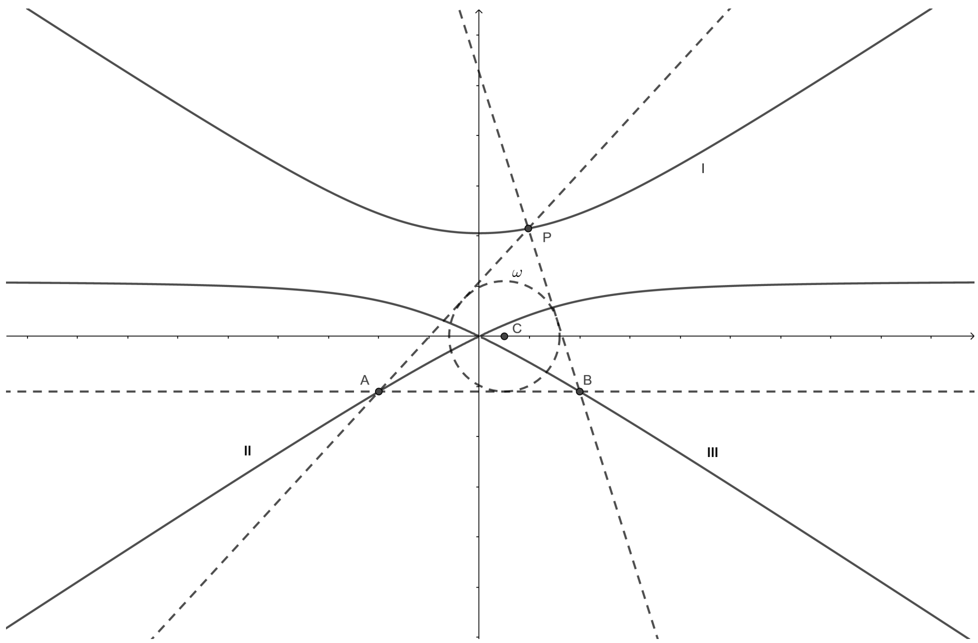

Figure 3: The cubic with $a = 2$ and $r = 1.1$ as locus of the third vertex of triangles with given side and incircle.

Analogously, we can find the equation of the tangent to $\omega(C, r)$ from B:

$$T_B : \; y + r = -\frac{2r(a-c)}{(a-c)^2 - r^2}(x - a) \; . \tag{2}$$

Let P be the intersection point of both tangents T_A and T_B, not lying on the circle. Its coordinates are solutions of the equations (1) and (2):

$$P = (x, y) = \left(\frac{c(a^2 - c^2 + r^2)}{a^2 - c^2 - r^2} \, , \, \frac{r(a^2 - c^2 + r^2)}{a^2 - c^2 - r^2} \right) \; .$$

After eliminating c from the coordinates, we find the following equation for the set of all points P:

$$(a^2 - r^2)y^3 - r(a^2 + r^2)y^2 - r^2yx^2 + r^3x^2 = 0 \; . \tag{3}$$

This irreducible cubic has three branches if $a > r$. The upper branch, labeled I in Figure 3, corresponds to the case where the circle is the incircle of the triangles. If $c^2 = a^2 - r^2$, then both tangents to $\omega(C, r)$ are parallel, and the point P goes over to the branches II and III if $c^2 > a^2 - r^2$.

If $a = r$, then the cubic only has two branches, and if $a < r$, i.e., the circle $\omega(C, r)$ is an externally tangent circle of the triangles, then the cubic becomes a strophoid-like curve as in Figure 4, in the limit $a \to 0$ a right strophoid.

The cubic has three asymptotes which are given by

$$A_1 : \qquad y = r \; ;$$

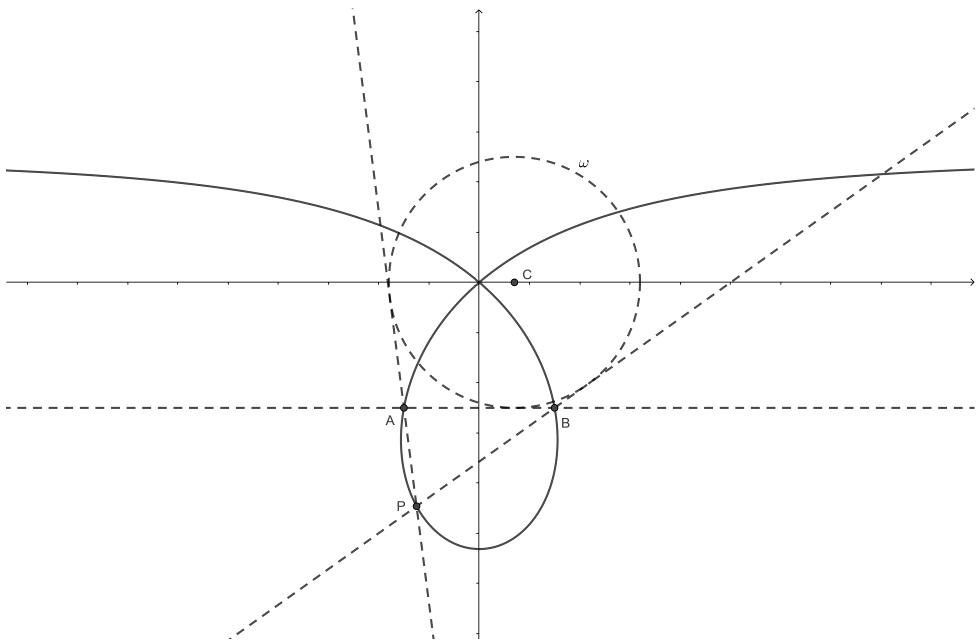

Figure 4: The cubic with $a = 1.5$ and $r = 2.5$ as locus of the third vertex of triangles with given side and excircle.

$$\begin{aligned} A_2 : \quad & r\sqrt{a^2 - r^2}x - (a^2 - r^2)y + r^3 = 0 \; ; \\ A_3 : \quad & r\sqrt{a^2 - r^2}x + (a^2 - r^2)y - r^3 = 0 \; . \end{aligned}$$

Note that the points A and B lie on the cubic. The cubic further has a crunode in the origin, i.e., the cubic crosses itself in $(0, 0)$ and has two distinct tangents there.

The classification of the cubics has a long history which goes back to Newton [1]. He found 72 different species, which were later extended to 78 by Stirling and Murdoch in the 18th century. The cubic in Figure 3 belongs to the 18th species as defined by Newton. This group of cubics was later called the Arachnida or spider curves [3, pp. 76–79]. In particular, if $a = \sqrt{8}$ and $r = 2$, then we recover precisely the Arachnida curve with parameter -3 as presented in [4]. For other values of a and r, with $a > r$, we obtain rescalings of the same Arachnida curve.

3 Generating the cubic from a conic

A major section in the study of plane curves is devoted to the generation of curves from other curves; see e.g. [5] for various methods. In this tradition, in [2] the author gave a universal construction method for cubic curves with a node using conics. Without going into the details of this procedure, we sketch the specific construction method for the cubic in (3).

Let $B_1 = (b, 0)$ and $B_2 = (2b, 0)$ be two points on the x-axis. Denote by $V = \left(0, \frac{r(a^2+r^2)}{a^2-r^2}\right)$

the vertex of the cubic and consider the two points

$$Q = \left(0, \frac{r(a^2+r^2)}{a^2}\right) \ , \ S = \left(\frac{(a^2+r^2)^{3/2}b}{a^2b+(a^2+r^2)^{3/2}}, \frac{r(a^2+r^2)b}{a^2b+(a^2+r^2)^{3/2}}\right) \ ,$$

whereby S lies on a tangent to the cubic at the node. We then construct the conic σ going through the node, B_2 and S, and which is tangent to the y-axis in the node and to the line QB_2:

$$\sigma : \ r^2x^2 + \frac{2ba^2r}{a^2+r^2}xy + (a^2+r^2)y^2 - 2br^2x = 0 \ . \tag{4}$$

The intersection points M_1 and M_2 of the conic σ with the line VB_1, when B_1 moves along the x-axis, describe the curve (3). This can be seen immediately by writing down the equation of the line VB_1, namely,

$$VB_1 : \ y = -\frac{r(a^2+r^2)}{(a^2-r^2)b}(x-b) \ ,$$

and solving for b:

$$b = \frac{r(a^2+r^2)x}{r(a^2+r^2)-(a^2-r^2)y} \ .$$

If we substitute this expression for b in the equation of the conic (4), we find the equation of the cubic (3). Note that the branch I of the cubic is described by hyperbolas (see Figure 5), while the branches II and III are partly described by hyperbolas and ellipses (see Figure 5 and 6).

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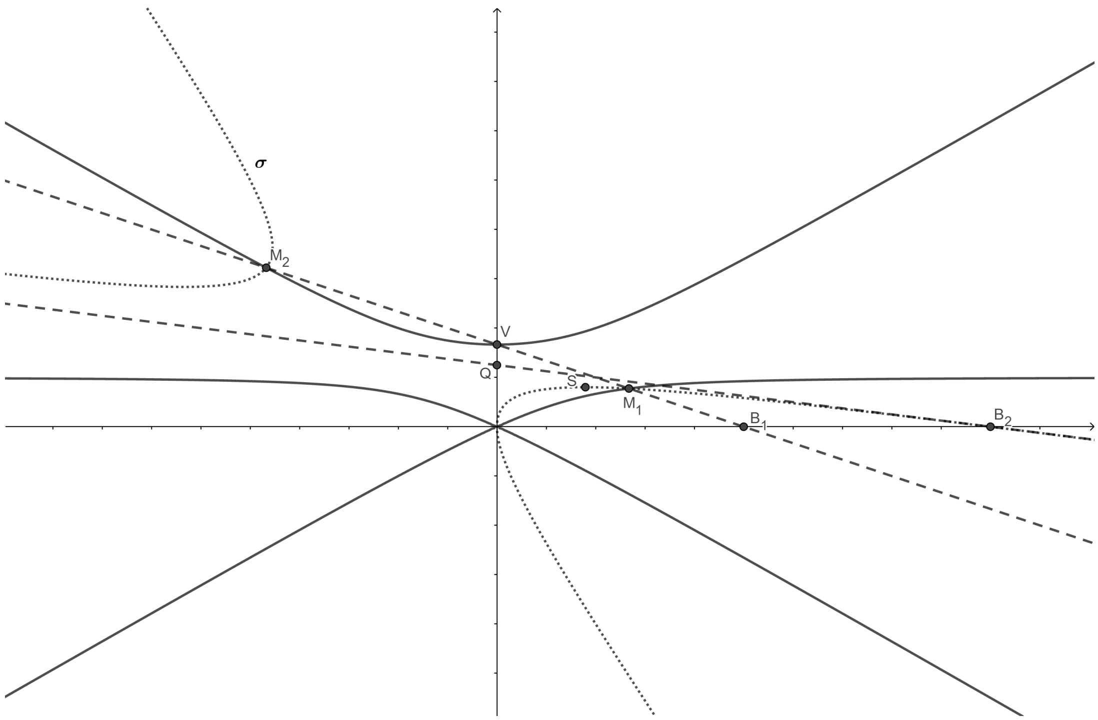

Figure 5: The cubic with $a = 2$ and $r = 1$ generated by a conic: a hyperbola with $b = 5$.

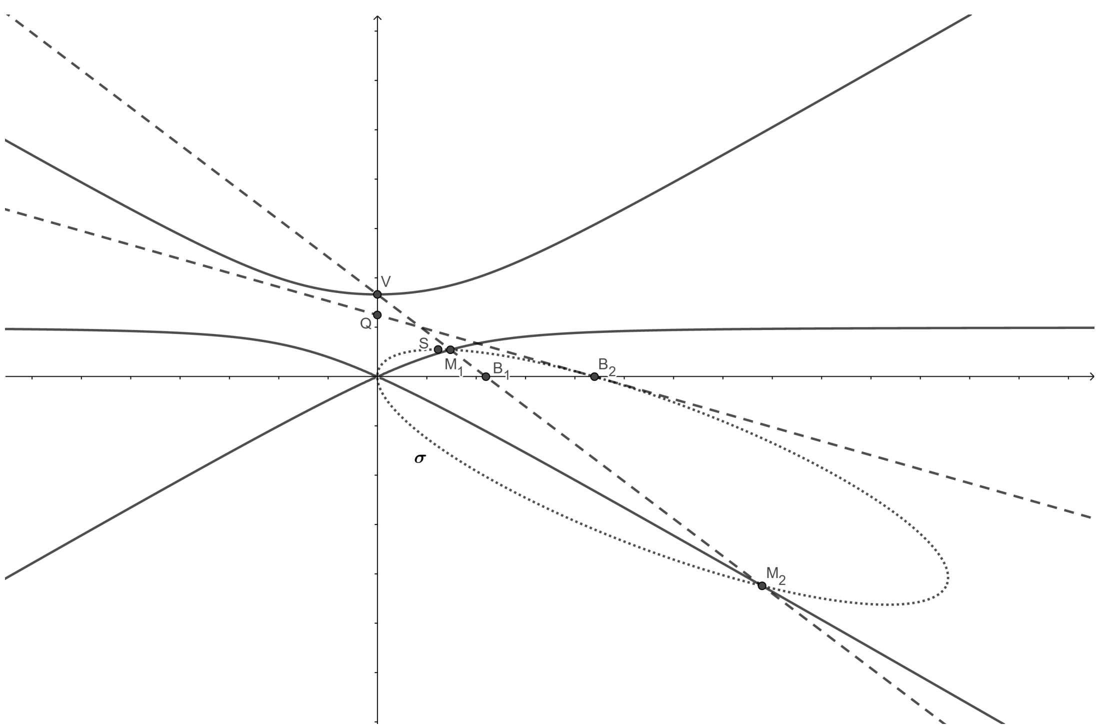

Figure 6: The cubic with $a = 2$ and $r = 1$ generated by a conic: an ellipse with $b = 2.2$.