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The ratio of perimeter to diameter of a regular polygon with infinite sides resulting in π

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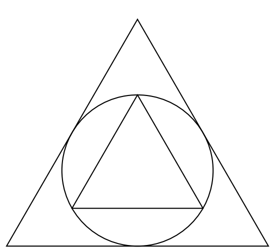
1 Introduction

For millennia, mathematicians have approximated π using equations of regular polygons. Among these mathematicians were Chinese mathematicians such as Liu Hui, and the Greek mathematician Archimedes.[1, 2].

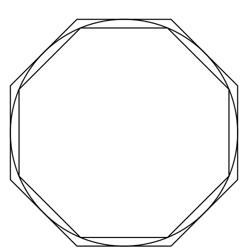
Hui knew that the value of π could be found by taking the circumference of a unit circle and dividing it by the diameter of a unit circle. He also knew that given the perimeter of any regular polygon, it was relatively easy to find the perimeter of a regular polygon with twice as many sides. Using this method, Hui was able to find the perimeters of polygons with increasingly large numbers of sides. As he found these perimeters, the regular polygons became closer and closer to circles, and therefore the ratio became closer and closer to the value of π. Hui used his method to approximate π using a 96-sided polygon and a 192-sided polygon, finding the value of π to be

$$3.141024 < \pi < 3.142708\,.$$

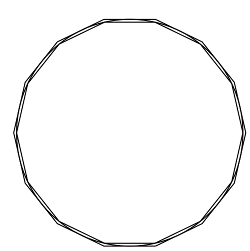
Archimedes used the perimeters of both inscribed and circumscribed polygons to approximate π. Like Hui, Archimedes knew that the value of π could be found by taking the circumference and dividing it by the diameter. By bounding the circle within inscribed and circumscribed polygons with an equal number of sides, Archimedes was able to bound the ratio. The figures shown below give a visual representation of this bounding. Below these figures is an algebraic representation of Archimedes' process, using the polygons provided in the figures.



3-sided regular polygon



8-sided regular polygon



14-sided regular polygon

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Using inscribed and circumscribed 3-sided regular polygons, the value of π is bounded between the ratios of inscribed and circumscribed polygons: $2.5981 < \pi < 5.1962$.

Using inscribed and circumscribed 8-sided regular polygons, the value of π is bounded between the ratios of inscribed and circumscribed polygons: $3.0615 < \pi < 3.3137$.

Using inscribed and circumscribed 14-sided regular polygons, the value of π is bounded between the ratios of inscribed and circumscribed polygons: $3.1153 < \pi < 3.1954$.

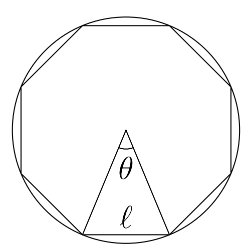
This process is extremely tedious, and Archimedes continued to use it for up to 96-sided regular polygons, approximating the value of π as

$$\frac{223}{71} < \pi < \frac{22}{7}, \quad \text{or} \quad 3.1408 < \pi < 3.1429\,.$$

In 1630, Austrian astronomer Christoph Grienberger took Archimedes' method to the next level, approximating π to 38 digits, using 10^{40}-sided polygons. This is the most accurate approximation achieved using this method. [3]

At the time of these methods, Archimedes and Hui did not have access to significant areas of math known to us today, such as calculus. Taking from these methods, it's reasonable to predict that taking the perimeter of a regular polygon with an infinite number of sides and dividing it by the diameter would result in π; you may have even thought of the idea yourself! This article presents a quick and easy method of finding π using both the methods of ancient mathematicians and basic calculus.

2 Finding π by geometry and calculus

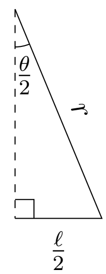


n-sided regular polygon

Given a circle of radius r, the perimeter P of its inscribed n-sided regular polygon is

$$P = \ell n$$

where ℓ is the length of each side. To express ℓ and thus P in terms of r and n, we can note that $\theta = \frac{360^\circ}{n}$ and then see from each of the triangles



that

$$\frac{\ell}{2} = r \sin\left(\frac{\theta}{2}\right) = r \sin\left(\frac{180°}{n}\right).$$

Therefore,

$$P = n\ell = 2rn \sin\left(\frac{180°}{n}\right).$$

The circumference of the circle is $2\pi r$ and is also equal to the limiting value of the circumference P as n grows to infinity:

$$2\pi r = \lim_{n\to\infty} P = \lim_{n\to\infty} 2rn \sin\left(\frac{180°}{n}\right) = 2r \lim_{n\to\infty} n \sin\left(\frac{180°}{n}\right),$$

so we have the formula

$$\pi = \lim_{n\to\infty} n \sin\left(\frac{180°}{n}\right).$$

By calculating $n \sin\left(\frac{180°}{n}\right)$ for increasing values of n, we get an increasingly precise value of π.

Note that, by replacing degrees by radians, we get the formula

$$\pi = \lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right).$$

This is of course not useful for calculating the value of π but it is still nice.

Having just taken my first calculus class when writing this, I recognized that I could prove that last expression by applying L'Hôpital's Rule:

Theorem 1 (L'Hôpital's Rule [5])**.** *Suppose that f and g are differentiable real functions on an open interval I containing a real number a such that $f(a) = g(a) = 0$ and that $g'(x) \neq 0$ for all $x \in I - \{a\}$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)}$$

if the latter limit exists.

Now applying L'Hôpital's Rule, we find that

$$\lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{n^{-1}} = \lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{n}\right)\left(-\pi n^{-2}\right)}{-n^{-2}} = \lim_{n\to\infty} \pi \cos\left(\frac{\pi}{n}\right) = \pi.$$

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