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Patterns and the many paths of problem solving

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1 Introduction

Consider the following three equations:

$$2^2 + 3^2 + 6^2 = 7^2$$
$$3^2 + 4^2 + 12^2 = 13^2$$
$$4^2 + 5^2 + 20^2 = 21^2\,.$$

Can you spot a pattern here?

Here are three other equations [1]:

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2\,.$$

Do they have a specific pattern?

Furthermore, look at the following pattern:

$$1^3 + 2^3 = (1+2)^2$$
$$1^3 + 2^3 + 3^3 = (1+2+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2\,.$$

What is the general pattern?

For the first set of equations above, we might guess that

Identity 1.

$$a^2 + (a+1)^2 + (a(a+1))^2 = (a(a+1)+1)^2$$

for all natural numbers a.

Similarly for the second set of equations above, we can predict that

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Identity 2.

$$\sum_{k=2n^2}^{2n^2+2n} k^2 = \sum_{k=2n^2+2n+1}^{2n^2+3n} k^2$$

for all natural numbers n*.*

Furthermore, for the third set of equations above, it is probably valid that

Identity 3.

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

for every natural number n*.*

The goal of our paper is to prove these three conjectures in multiple ways, namely using algebraic and geometric methods. We will show that two quantities are equal if they can be manipulated into the same forms using algebraic operations. We will then represent quantities using geometric shapes. We represent addition by combining shapes and represent subtraction by removing shapes. We also visualize multiplication by the area of a rectangle while division as partitioning a rectangle into equal parts. Finally, we show equality of two collections of objects by proving that both collections can be arranged into two congruent objects without gaps and overlaps.

2 Pattern 1

Proof with Algebra

We calculate:

$$\begin{aligned} a^2 + (a+1)^2 + (a(a+1))^2 &= a^2 + (a^2 + 2a + 1) + (a^4 + 2a^3 + a^2) \\ &= a^4 + 2a^3 + 3a^2 + 2a + 1 \\ &= a^2(a^2 + 2a + 1) + 2a(a+1) + 1 \\ &= (a(a+1))^2 + 2a(a+1) + 1 \\ &= (a(a+1)+1)^2 . \end{aligned} \qquad \square$$

Proof with Pictures

Figure 1 shows that three squares with areas $a(a+1)^2$, a^2 and $(a+1)^2$ that together form a square whose area is $((a+1)+1^2)$. Accordingly,

$$a^2 + (a+1)^2 + (a(a+1))^2 = (a(a+1)+1)^2 . \qquad \square$$

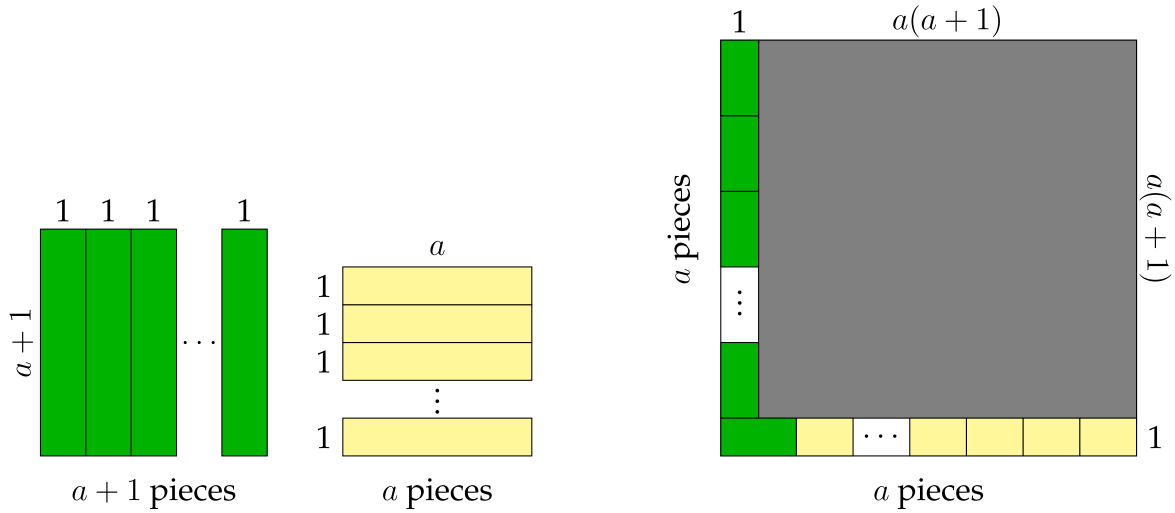


Figure 1: Three squares with areas $a(a+1)^2$, a^2 and $(a+1)^2$ forming a square with area $((a+1)+1^2)$.

3 Pattern 2

The second type of equations start with the numbers

$$3, 10, 21, 36, 55, \dots .$$

The successive differences between these numbers are

$$7, 11, 15, 19, \dots ,$$

which appears to form the linear sequence $7 + 4n$ for $n = 1, 2, \dots$. If this is true, then the numbers $3, 10, 21, \dots$ form a quadratic sequence $an^2 + bn + c$ for $n = 1, 2, \dots$. This sequence has values 3, 10 and 21 for $n = 1, 2, 3$.

We can solve the equations

$$\begin{aligned} a + b + c &= 3 \\ 4a + 2b + c &= 10 \\ 9a + 3b + c &= 21 \end{aligned}$$

to find that $a = 2$, $b = 1$ and $c = 0$. The first number in the nth equation is therefore $2n^2 + n$.

We also can see from the pattern that for the n-th line, there are $n + 1$ terms on the left hand side and there are n terms on the right hand side. Furthermore, we can see that the terms on each line are consecutive integers. We finally conclude that the equations have the following form:

$$\sum_{k=2n^2+n}^{2n^2+2n} k^2 = \sum_{k=2n^2+2n+1}^{2n^2+3n} k^2 \,.$$

for all natural numbers n.

Alternatively, we can prove the above equation by showing that

$$\sum_{k=2n^2+2n+1}^{2n^2+3n} k^2 - \sum_{k=2n^2+n+1}^{2n^2+2n} k^2 = (2n^2+n)^2 \,.$$

Furthermore, let $a = 2n^2 + n$; then the equation above can be simplified to

$$\sum_{k=a+n+1}^{a+2n} k^2 - \sum_{k=a+1}^{a+n} k^2 = a^2 \,.$$

Proof with Algebra

We will prove that for every natural number n and for $a = 2n^2 + n$,

$$\sum_{k=a+n+1}^{a+2n} k^2 - \sum_{k=a+1}^{a+n} k^2 = a^2 \,.$$

We can see that

$$\begin{aligned}
\sum_{k=a+n+1}^{a+2n} k^2 - \sum_{k=a+1}^{a+n} k^2 &= (a+n+1)^2 + (a+n+2)^2 + \cdots + (a+2n-1)^2 + (a+2n)^2 \\
&\quad - \big((a+n)^2 + (a+n-1)^2 + \cdots + (a+2)^2 + (a+1)^2\big) \\
&= (2a+2n+1)\big(1+3+\cdots+(2n-3)+(2n-1)\big) \\
&= (2a+2n+1)n^2 \\
&= (4n^2+2n+2n+1)n^2 \\
&= 4n^4+4n^3+n^2 \\
&= (2n^2+n)^2 \\
&= a^2 \,.
\end{aligned}$$

□

Proof with Pictures

We will prove that for every natural number n and for $a = 2n^2 + n$,

$$\sum_{k=a+n+1}^{a+2n} k^2 - \sum_{k=a+1}^{a+n} k^2 = a^2 \,.$$

We first define $A_i = (a+n+i)^2$ and $B_i = (a+n+1-i)^2$ for each $i = 1, \ldots, n$, and note that

$$\sum_{k=a+n+1}^{a+2n} k^2 = \sum_{i=1}^{n} A_i \qquad \text{and} \qquad \sum_{k=a+1}^{a+n} k^2 = \sum_{i=1}^{n} B_i\,.$$

We place B_1 on the corner of A_1; see Figure 2a. Figure 2b shows that the remaining green area can be rearranged into a rectangle with height

$$(a+n) + (a+n+1) = 2a + 2n + 1 = b\,.$$

Notice that the use of b here is for convenience.

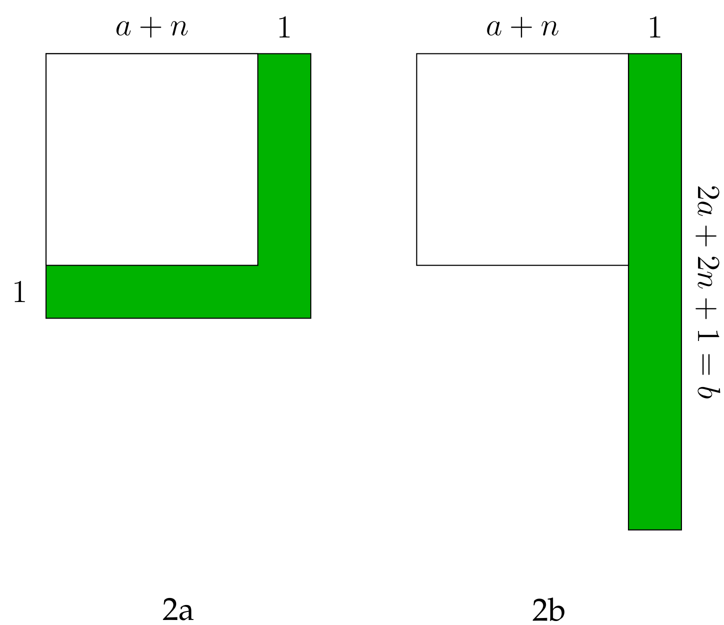


Figure 2: Decomposing A_1 from B_1

We apply the same procedure to each combination of $A_i - B_i$. Then, we arrange the resulting rectangles according to Figure 3a if n is even and according to Figure 3b if n is odd.

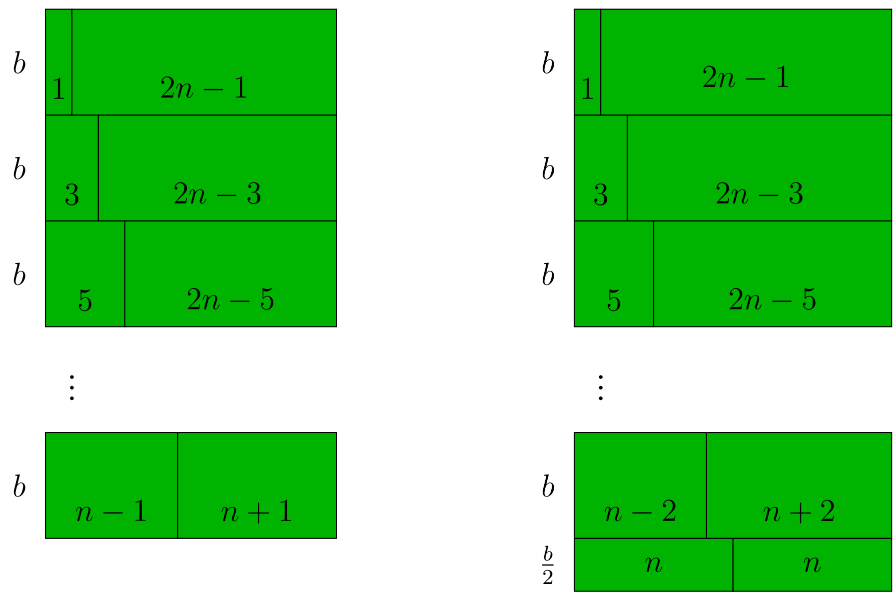
We can see that any value of n will form a rectangle with a width of $2n$ and a height of $\frac{n}{2}b$. Accordingly, the area of the rectangle formed is

$$\frac{n}{2}b(2n) = bn^2 = (2a + a/n)n^2 = 2an^2 + an = a(2n^2 + n) = a^2\,.$$

Therefore,

$$\sum_{k=a+n+1}^{a+2n} k^2 - \sum_{k=a+1}^{a+n} k^2 = a^2\,.$$

□



3a 3b

Figure 3: Arranging decomposed $A_i - B_i$ figures into a rectangle.

4 Pattern 3

4.1 Proof with Algebra

We will prove that for every natural number n,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

using mathematical induction. For $n = 2$,

$$1^3 + 2^3 = 1 + 8 = 9 = 3^2 = (1 + 2)^2 .$$

Now, assume that, for some natural number $i \geq 2$,

$$1^3 + 2^3 + \cdots + i^3 = (1 + 2 + \cdots + i)^2 .$$

Then

$$\begin{aligned} 1^3 + 2^3 + \cdots + i^3 + (i+1)^3 &= (1 + 2 + \cdots + i)^2 + (i+1)^3 \\ &= \left(\frac{i(i+1)}{2}\right)^2 + (i+1)^3 \\ &= \frac{(i^4 + 2i^3 + i^2) + 4(i^3 + 3i^2 + 3i + 1)}{4} \\ &= \frac{i^4 + 6i^3 + 13i^2 + 12i + 4}{4} \end{aligned}$$

$$= \left(\frac{(i+1)(i+2)}{2}\right)^2$$
$$= \left(1+2+\cdots+i+(i+1)\right)^2.$$

Therefore,

$$1^3 + 2^3 + \cdots + i^3 + (i+1)^3 = (1+2+\cdots+i+(i+1))^2.$$

and the proof follows by induction. □

4.2 Proof with Pictures

We will prove that for every natural number n,

$$1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$$

using pictures.

One way to prove the equality between squares and cubes is to consider the squares as the base area of a cuboid with a height of 1 unit. By the formula, the volume of a cuboid whose base area is $(1+2+\cdots+n)^2$ and height is 1 is $(1+2+\cdots+n)^2$. We will show that such cuboids can be arranged into a collection of cubes with sides $1, 2, \ldots, n$.

For $n = 5$, we can illustrate the base of the cuboid as follows:

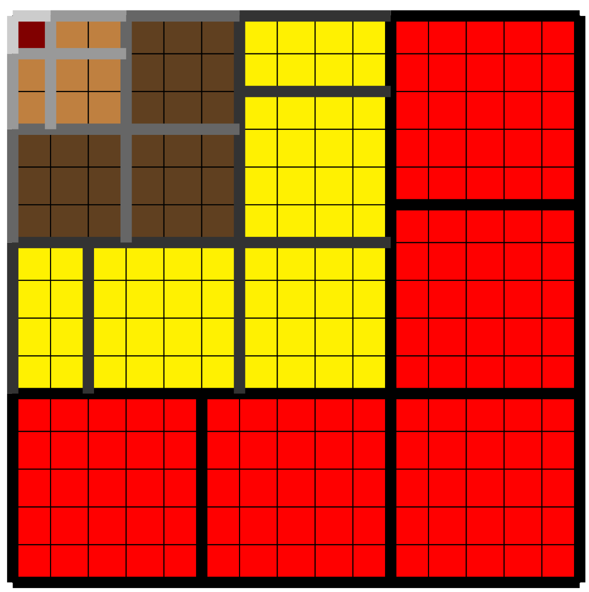


Figure 4: Partition of the base of a cuboid with a dimension of $(1+2+3+4+5)^2$

We can see that we have one layer of cuboids with a base area of 1^2, resulting in a cube of volume 1^3. We also have two layers of cuboids with a base area of 2^2, resulting in a cube of volume 2^3, and so on, until we have five layers of cuboids with a base area of 5^2, resulting in a cube of volume 5^3. We can continue the pattern for any natural number n. Accordingly,

$$1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2.$$ □

5 Afterthought

More General Values?

We have proven the some identities both algebraically and geometrically. For Identity 1, it is true for all natural numbers a. Is it possible to change a into a non-natural number? If so, is the identity still true if a is a fraction, or negative, or the more challenging, complex? How about the value of n in Identity 2 and Identity 3?

Other Scenarios?

We can rewrite Identity 1 as follows:

$$a^2 + (a+1)^2 = \big(a(a+1)+1\big)^2 - \big(a(a+1)\big)^2 .$$

In other words, the sum of the square of two consecutive natural numbers can be written as the difference between the square of two other consecutive natural numbers. It turns out that many other pairs of squares can be written as the difference between the square of two consecutive natural numbers. Under what condition do a pair of squares do not possess the property? How do you visualize your answer?

Next, how can you use Identity 3 to find a shortcut to calculate

$$2^3 + 4^3 + 8^3 + \cdots + (2n)^3$$

and

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3\,?$$

More Patterns?

Let us observe another pattern. Let $t(n)$ be the number of divisors of a natural number n. For example, $t(1) = 1, t(4) = 3, t(12) = 6$, and $t(p) = 2$ for any prime p.

Furthermore, let $\sum_{d|n} f(d)$ denote the sum of all divisors of n. For example,

$$\sum_{d|12} f(d) = f(1) + f(2) + f(3) + f(4) + f(6) + f(12) .$$

Let us observe the following interesting calculation:

$$\sum_{d|6} t(d)^3 = t(1)^3 + t(2)^3 + t(3)^3 + t(6)^3 = 1^3 + 2^3 + 2^3 + 4^3 = 81 = (1+2+2+4)^2 = \Big(\sum_{d|6} t(d)\Big)^2 .$$

In addition, for any prime p, we can see that

$$\sum_{d|p} t(d)^3 = t(1)^3 + t(p)^3 = 1^3 + 2^3 = 9 = (1+2)^2 = \Big(\sum_{d|p} t(d)\Big)^2 .$$

In fact, for any prime power p^a, using Identity 3 we have proven that

$$\begin{aligned}\sum_{d|p^a} t(d)^3 &= t(1)^3 + t(p)^3 + t(p^2)^3 + \cdots + t(p^a)^3 \\ &= 1^3 + 2^3 + 3^3 + \cdots + (a+1)^3 \\ &= (1 + 2 + 3 + \cdots + (a+1))^2 \\ &= (t(1) + t(p) + t(p^2) + \cdots + t(p^a))^2 \\ &= \Big(\sum_{d|p^a} t(d)\Big)^2.\end{aligned}$$

Now, we conclude that for some values of n,

$$\sum_{d|n} t(d)^3 = \Big(\sum_{d|n} t(d)\Big)^2.$$

Can you find all possible values of n such that the equation above is valid? In what ways can we visualize this pattern?

Now, let us try the other way around. Look at the following picture:

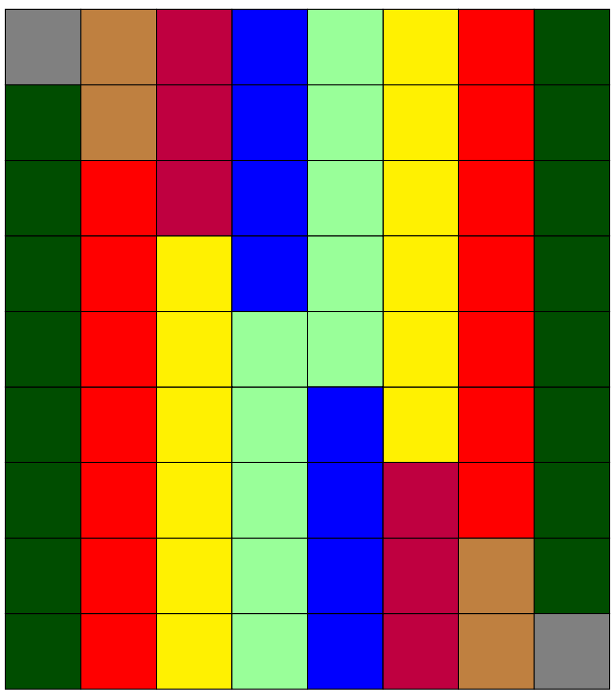


Figure 5: Partition of the base of a prism with a dimension of $(1+2+3+4+5)^2$

What algebraic identity can we come up from the given picture?

Acknowledgements

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References

[1] G. Mackiw, A combinatorial approach to sums of integer powers, *Mathematics Magazine* **73 (1)** (2000), 44–46.