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The harmonic series and its close friends: A dive into the Kempner series

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Introduction

Within a standard calculus class, convergent and divergent series are extremely important, but students do not learn what happens when series are slightly changed. By changing the harmonic series, in [2] during the 1910s, A.J. Kempner proved the convergence of such a series by removing every term that included the digit 9. We seek to do the same thing here but generalize the result to all bases.

Preliminary Definitions and Theorems

The next few definitions and theorems are background knowledge and notation required to understand the ideas established in the body of the paper.

Definition 1. Let

$$\sum_{n=0}^{\infty} s_n = s_0 + s_1 + s_2 + \cdots$$

be some series and define $S_i = \sum_{n=0}^{i} = s_0 + s_1 + \cdots + s_i$ to be the ith partial sum of this series. This series is said to *converge* to a value C if and only if the limit of partial sums, $\lim\limits_{i\to\infty} S_i$, exists. In particular, this limit equals C. Conversely, if $\lim\limits_{i\to\infty} S_i$ does not exist or equals infinity, then we say the series *diverges*.

Definition 2. A *geometric series* is any series of the form $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$.

Theorem 3. *The geometric series converges to* $\dfrac{a}{1-r}$ *when* $|r| < 1$ *and diverges when* $|r| \geq 1$.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} ar^n = a\frac{1-r}{1-r}\sum_{n=0}^{\infty} r^n &= \frac{a}{1-r}\left(\sum_{n=0}^{\infty} r^n - \sum_{n=0}^{\infty} r^{n+1}\right)\\ &= \frac{a}{1-r}\left(\sum_{n=0}^{\infty} r^n - \sum_{n=1}^{\infty} r^n\right) = \frac{a}{1-r}r^0 = \frac{a}{1-r}\,.\end{aligned}$$

□

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Definition 4. The *harmonic series* is the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$.

Theorem 5. *The harmonic series diverges.*

Proof.

$$\sum_{n=0}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots . \qquad \square$$

Definition 6. The *p-series* is defined as $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$.
Note that when $p = 1$, we get the harmonic series.

Theorem 7. *The* p*-series diverges when* $p \leq 1$ *and converges when* $p > 1$.

For a proofs of Theorem 7, we point the reader to the textbook [1].

Definition 8. Let b denote a base, x a digit, and $i \in \mathbb{N}$. Define the set

$$S_{(x,b,i)} = \{\text{Numbers not containing the digit } x \text{ in base } b \text{ in the interval } [b^i, b^{i+1})\}.$$

Note that the numbers in this set have $i+1$ digits. Also define

$$S_{(x,b)} = \{\text{Numbers not containing the digit } x \text{ in base } b\}.$$

Now we're all set to present the main results.

The harmonic series with certain terms deleted

Digit 9 deleted in base 10

Here we are working with the set $S_{(9,10)}$ and note the cardinality of this set.

Proposition 9. $|S_{(x,10,i)}| = 8 \cdot 9^i$ *for each nonzero digit* x.

Proof. Let $n \in \mathbb{N}$ be a number with $i+1$ digits. For the leftmost digit of n, we have 8 choices since we can't choose 0 or x. For the other i digits, we have 9 choices since we can't choose x. Multiplying these numbers gives $8 \cdot 9^i$. $\square$

The cardinality $|S_{(9,10,i)}| = 8 \cdot 9^i$ turns out to be important for determining the convergence of *Kempner series* which is the series of inverses of all positive integers that do not contain 9 as a digit. This series who first studied in the 1910s, by Kempner [2] who proved that this series converges:

Proposition 10. $\sum_{n \in S_{(9,10)}}^{\infty} \frac{1}{n}$ *converges.*

It's interesting that the series of the deleted terms (containing 9 among their digits) diverges since this sum is strictly greater than $\sum_{k=0}^{\infty} \frac{1}{10k+9} > \sum_{k=0}^{\infty} \frac{1}{10k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{1}{k}$ which diverges. Oddly, the difference between the harmonic series and the series of deleted terms is a convergent series.

Proof. We can write

$$\sum_{n \in S_{(9,10)}}^{\infty} \frac{1}{n} = \overbrace{1 + \cdots + \frac{1}{8}}^{|S_{(9,10,1)}|=8} + \underbrace{\frac{1}{10} + \cdots + \frac{1}{18} + \frac{1}{20} + \cdots + \frac{1}{88}}_{|S_{(9,10,2)}|=89} + \cdots .$$

Note that

$$\overbrace{1 + \cdots + \frac{1}{8}}^{|S_{(9,10,1)}|=8} < 1 + \cdots + 1 = 8\,;$$
$$\overbrace{\frac{1}{10} + \cdots + \frac{1}{88}}^{|S_{(9,10,2)}|=89} < \frac{1}{10} + \cdots + \frac{1}{10} = \frac{89}{10}$$

and, in general for $i \geq 1$, that

$$\overbrace{\frac{1}{10^i} + \cdots + \frac{1}{M}}^{|S_{(9,10,i)}|=8 \cdot 9^i} < \frac{1}{10^i} + \cdots + \frac{1}{10^i} = \frac{8 \cdot 9^i}{10^i} .$$

where $M = \max S_{(9,10,i)}$. By these upper bounds and by Theorem 3, we see that

$$\sum_{n \in S_{(9,10)}}^{\infty} \frac{1}{n} < \sum_{i=0}^{\infty} 8\left(\frac{9}{10}\right)^i = \frac{8}{1 - \frac{9}{10}} = 80 < \infty\,,$$

so the series converges. $\square$

By modifying the above proof slightly, we can generalise Proposition 10 as follows.

Theorem 11. *For each nonzero digit x,* $\sum_{n \in S_{(x,10)}}^{\infty} \frac{1}{n} < 80$*. In particular,* $\sum_{n \in S_{(x,10)}}^{\infty} \frac{1}{n}$ *converges.*

In the 1970s, Baille [3] approximated this Kempner series in Theorem 11 to around 22.92 and continuing on this problem more recently in 2008, Baille and Schmelzer [4] found an efficient algorithm for any string of omitted digits.

Deleting a digit in base b

We now generalise the above results to all bases $b \geq 2$.

Proposition 12. $|S_{(x,b,i)}| = (b-2)(b-1)^i$.

Proof. For each base b $(i+1)$-digit number $n \in \mathbb{N}$, there are $b-2$ choices for the first digit and $b-1$ choices for the other i digits. □

Theorem 13. $\displaystyle\sum_{n \in S_{(x,b)}}^{\infty} \frac{1}{n} < (b-2)b$. *In particular,* $\displaystyle\sum_{n \in S_{(x,b)}}^{\infty} \frac{1}{n}$ *converges.*

Proof. By Proposition 12 and Theorem 3,

$$\sum_{n \in S_{(x,b)}}^{\infty} \frac{1}{n} = \sum_{i=1}^{\infty} \sum_{n \in S_{(x,b,i)}}^{\infty} \frac{1}{n} < \sum_{i=1}^{\infty} |S_{(x,b,i)}| \frac{1}{b^i}$$

$$= \sum_{i=1}^{\infty} (b-2) \frac{b-1}{b^i} = \sum_{i=1}^{\infty} (b-2) \left(\frac{b-1}{b}\right)^i = (b-2)b\,. \qquad \square$$

p-series with certain terms deleted

This section expands the previous results for harmonic series.

Theorem 14. $\displaystyle\sum_{n \in S_{(x,b)}}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > \log_b(b-1)$.

Proof. For each integer $i \geq 1$, define

$$A_i = \sum_{n \in S_{(x,b,i)}} \frac{1}{n^p}\,.$$

By Proposition 12,

$$\frac{(b-2)(b-1)^i}{(b^p)^{i+1}} < A_i < \frac{(b-2)(b-1)^i}{(b^p)^i}$$

since $\frac{1}{b^{pi}}$ is the largest term in A_i and $\frac{1}{(b^p)^{i+1}}$ is strictly less than all terms in A_i. Therefore,

$$\sum_{i=0}^{\infty} \frac{(b-2)(b-1)^i}{(b^p)^{i+1}} < \sum_{i=0}^{\infty} A_i < \sum_{i=0}^{\infty} \frac{(b-2)(b-1)^i}{(b^p)^i}\,.$$

Simplifying further gives us

$$\frac{b-2}{b^p} \sum_{i=0}^{\infty} \left(\frac{b-1}{b^p}\right)^i < \sum_{n \in S_{(x,b,i)}} \frac{1}{n^p} < (b-2) \sum_{i=0}^{\infty} \left(\frac{b-1}{b^p}\right)^i.$$

Therefore by Theorem 3, the series $\sum_{n \in S_{(x,b,i)}} \frac{1}{n^p}$ converges if and only if $\frac{b-1}{b^p} < 1$; that is, when $p > \log_b(b-1)$. □

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