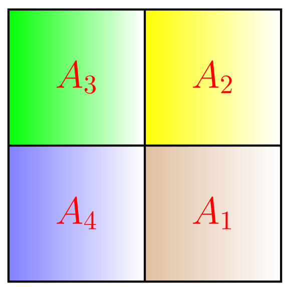
Parabola Volume 57, Issue 3 (2021)

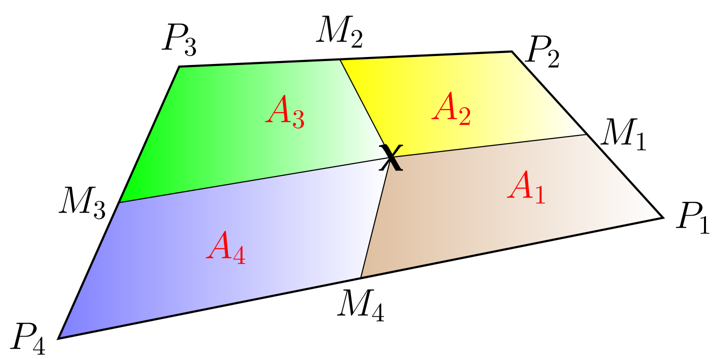
Solutions 1651–1660

Problems 1651–1660 are dedicated to the editor of *Parabola*, Thomas Britz, and his partner Ania, in celebration of the arrival of their twin sons Alexander and Benjamin.

Q1651 To celebrate Alexander and Benjamin's 4–month "birthday", Thomas decided to bake a square cake and share it equally among the twins by cutting from the centre of the square to the midpoint of each side, and giving pieces A_1, A_3 to one twin and A_2, A_4 to the other.

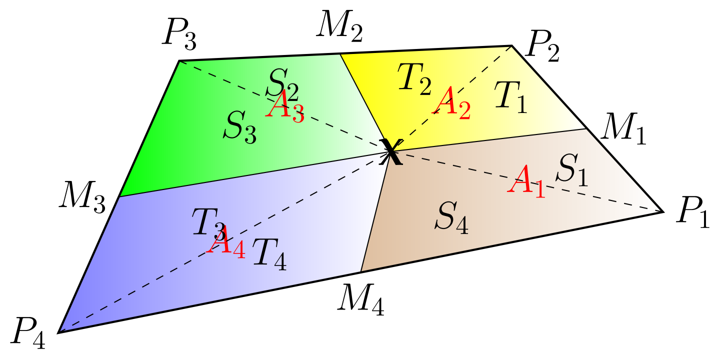


Unfortunately, when the cake came out of the oven, it wasn't exactly square...



However, the twins' mother Ania was able to assure Thomas that cutting the cake in the same way (from some point to the midpoint of each edge, as before) would still give equal shares to Alexander and Benjamin. Prove that she was correct.

SOLUTION We wish to show that in any (convex) quadrilateral $P_1P_2P_3P_4$, if X is a point inside the quadrilateral and M_1, M_2, M_3, M_4 are the midpoints of P_1P_2, P_2P_3, P_3P_4, P_4P_1 respectively, then the total area of Alexander's pieces $XM_1P_2M_2$ and $XM_3P_4M_4$ is equal to the total area of Benjamin's pieces $XM_2P_3M_3$ and $XM_4P_1M_1$.



Now Alexander's share consists of the triangular areas T_1, T_2, T_3, T_4, and Benjamin's share consists of S_1, S_2, S_3, S_4. However, triangles $\triangle XM_1P_1$ and $\triangle XM_1P_2$ have the same bases (because M_1 is the midpoint of P_1P_2) and the same altitudes, so they have the same area: that is, $S_1 = T_1$. By the same argument, $S_2 = T_2$, $S_3 = T_3$ and $S_4 = T_4$, and therefore the total amount of cake is the same for each twin.

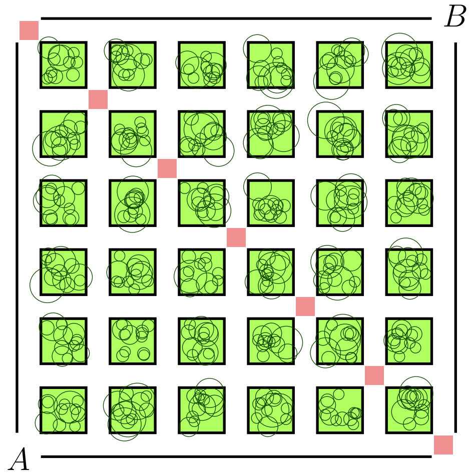
Q1652 Alexander and Benjamin meet a girl called Christine, who tells them that she has a twin sister Denise.
What is the probability that Christine and Denise are identical twins?

Assume the following figures: 30% of all twins are identical, 70% are non–identical; identical twin pairs are equally likely to be boy–boy or girl–girl (boy–girl is impossible); of non–identical twins 30% are boy–boy, 30% are girl–girl and 40% are boy–girl.

SOLUTION In the entire population of twins, a proportion 0.5×0.3 are identical girls and 0.3×0.7 are non–identical girls. Since Christine says she has a twin sister, she must be in one of these categories; and the probability that she is in the former is

$$\frac{0.5 \times 0.3}{0.5 \times 0.3 + 0.3 \times 0.7} = \frac{5}{12}\,.$$

Q1653 Alexander and Benjamin are playing in their local park. This park consists of an n by n array of square gardens, separated by paths. Alexander starts at the south–west corner of the park and walks along the paths at a speed which takes him along the side of any garden square in exactly one minute, and always heads north or east. Benjamin walks at the same speed, but starts at the north–east corner and always walks south or west. Find the probability that Alexander and Benjamin meet after n minutes.



SOLUTION After n minutes, each of the twins will be at some point on the north–west/south–east diagonal of the park; that is, at one of the intersections marked in red. We wish to find the probability that they are at the same intersection. Now if we set up a coordinate system such that Alexander starts at $(0,0)$ and Benjamin starts at (n,n), then the diagonal points have coordinates $(k,n-k)$ for $k=0,1,2,\ldots,n$. Alexander's trip from $(0,0)$ to $(k,n-k)$ must consist of k steps in an easterly direction and $n-k$ in a northerly direction, and can be specified by a list such as $NNEN\cdots NNNE$. The number of such lists is given by the binomial coefficient $C(n,k)$. The number of ways in which Benjamin can reach $(k,n-k)$ is exactly the same. And the total number of possibilities for $2n$ choices of direction (n for each twin) is 2^{2n}. Since k can be any integer from 0 to n, the required probability is

$$p=\frac{1}{2^{2n}}\big(C(n,0)^2+C(n,1)^2+C(n,2)^2+\cdots+C(n,n)\big)\ .$$

We can simplify the sum in brackets by considering the coefficient of x^n in the binomial expansion of

$$(x+1)^n(1+x)^n=(x+1)^{2n}\ .$$

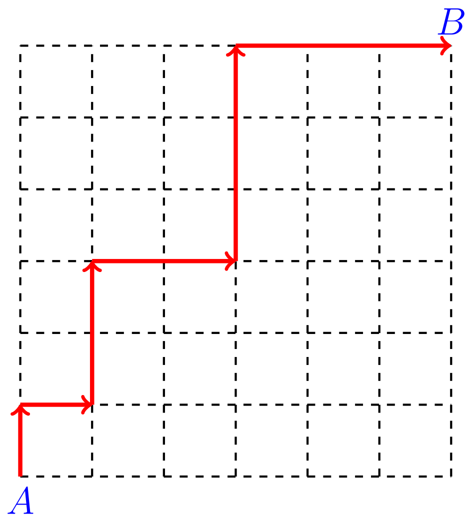
A term in x^n on the left hand side is obtained as a product of terms in x^k1^{n-k} and 1^kx^{n-k}; the coefficient is $C(n,k)C(n,k)=C(n,k)^2$. Any k from 0 to n gives a product of x^n, so the total coefficient is

$$C(n,0)^2+C(n,1)^2+C(n,2)^2+\cdots+C(n,n)^2\ .$$

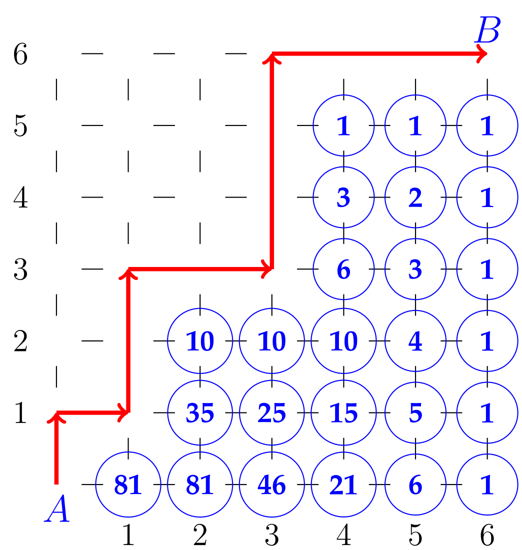
But the coefficient of x^n in $(x+1)^{2n}$ is $C(2n,n)$, which therefore equals this sum. Hence the required probability is

$$\frac{C(2n,n)}{2^{2n}}\ .$$

Q1654 Alexander and Benjamin are walking in the park again, and we can now reveal that in actual fact the number of gardens is 6×6. If Alexander walks from A to B along the path shown in red, in how many ways can Benjamin walk from B to A without meeting Alexander's path (except at the beginning and end of course)? As in the previous problem, Benjamin only walks in a southerly or westerly direction.



SOLUTION It is clear that Benjamin's first walk must be to the south and that there is only one way to do it. We indicate this by writing a 1 on the $(6,5)$ intersection. Since Benjamin can only reach a path intersection from the north or from the east, the number of ways to get to each intersection is obtained by adding the number of ways of reaching the intersections immediately to the north and east, provided that these do not lie outside the park or on Alexander's red path. So we can fill in the diagram step by step to give the final answer of 81 ways.



Q1655 Alexander and Benjamin are visiting the nation of Twinnia. In this country there is a rule that on any given day, twins must behave alike in terms of telling the truth: that is, both must tell the truth or both must lie; it is forbidden for one to tell the truth and the other to lie. You overhear a conversation between four people. Two of them are Alexander and Benjamin, but you cannot decide which is which, though one of them is wearing a yellow jumper and one is wearing a red jumper. The other two are Ellie and Fiona: they look very similar, and you are not sure whether or not they are twins. The following statements are made.

Ellie: Fiona and I are twins.

Fiona: the boy in the yellow jumper is Benjamin.

Boy in yellow: the boy in the red jumper is Alexander.

Boy in red: Ellie and Fiona are not twins.

Can you determine which of the boys is which? Can you decide whether Ellie and Fiona are twins or not?

SOLUTION If Ellie spoke the truth, then she and Fiona are twins, so that Fiona must also have spoken the truth. Therefore the boy in yellow is Benjamin, and he spoke the truth when he said that the other boy was Alexander. Since Alexander is Benjamin's twin, he also spoke the truth, and so Ellie and Fiona are not twins. This contradicts what we discovered earlier; so the situation is impossible, and we must conclude that Ellie did not speak the truth.

Since Ellie lied and the boy in red contradicted her, he must have told the truth, and since the boy in yellow is his twin, he also spoke the truth. Therefore Alexander is wearing red and Benjamin is wearing yellow. Moreover, this means that Fiona spoke the truth while Ellie lied, and so they cannot be twins.

Q1656 Primes which differ by 2 are called **twin primes**. Prove that, with two exceptions, if a and b are twin primes then the last digit of $(a+b)(a^2+b^2)$ is 0.

SOLUTION Two examples of twin primes are $a=3, b=5$ and $a=5, b=7$; in these cases we calculate

$$(a+b)(a^2+b^2) = 272 \text{ or } 888$$

which do not end in 0, and these are the two exceptions referred to in the question. Now the last digit of a prime, other than 2 or 5, cannot be $0,2,4,5,6,8$. Since twin primes differ by 2, the last digits of a,b are $1,3$ or $7,9$ or $9,1$, or *vice versa*. In these cases, the last digits of $a+b$ and of a^2+b^2 are respectively

$$4 \text{ and } 0\,; \quad \text{or} \quad 6 \text{ and } 0\,; \quad \text{or} \quad 0 \text{ and } 2\,;$$

and in every case, the last digit of the product is 0.

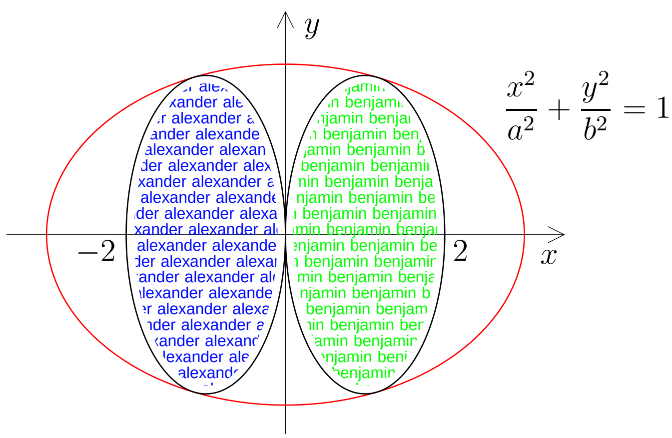
Q1657 Thomas is designing a new nursery for his twins Alexander and Benjamin. They will each have a cradle in the shape of an ellipse, placed side by side. In suitable coordinates, the ellipses have equations

$$(x+1)^2 + \frac{y^2}{4} = 1 \quad \text{and} \quad (x-1)^2 + \frac{y^2}{4} = 1 \, .$$

The cradles will be surrounded by a wooden floor. As a mathematician, Thomas is very keen on symmetry, so the surround will also be an ellipse, in this case having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, ;$$

but he is also conscious of using space efficiently, so he wants this ellipse to have the smallest possible area. What values of a and b should he choose?



SOLUTION It is clear from the diagram that the "boundary ellipse" should be tangent to each of the "cradle ellipses"; by symmetry, we only need to consider the tangency in the first quadrant. By calculus, the gradient of the tangent to the boundary ellipse is given by

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

and so

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y} \, ,$$

provided that $y \neq 0$. (This formula can also be determined without calculus, if you wish.) Suppose that the two ellipses meet with a common tangent at (p, q), and assume initially that $q \neq 0$. Then we have

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \, , \quad (p-1)^2 + \frac{q^2}{4} = 1 \, , \quad -\frac{b^2}{a^2}\frac{p}{q} = -\frac{4}{1}\frac{p-1}{q} \, .$$

The third equation is easily solved to give

$$p = \frac{4a^2}{4a^2 - b^2} \, .$$

Taking b^2 times the first equation minus 4 times the second eliminates q, giving

$$\frac{b^2p^2}{a^2} - 4(p-1)^2 = b^2 - 4\ ;$$

and we can now substitute the preceding expression for p into this. Some careful algebra results in an unexpectedly simple equation

$$b^4 - 4a^2b^2 + 16a^2 = 0\ ,$$

which can be treated as a quadratic in b^2 and solved to give

$$b^2 = 2a\big(a \pm \sqrt{a^2-4}\big)\ .$$

We must have $a > 2$, because $a = 2$ would correspond to $q = 0$, which we have ruled out. If we take the $+$ sign in this formula we have $b^2 > 2a^2$ and hence

$$p = \frac{4a^2}{4a^2 - b^2} > \frac{4a^2}{2a^2} = 2\ ,$$

which is impossible; so we must take the $-$ sign. Thus

$$b^2 = 2a\big(a - \sqrt{a^2-4}\big)\ .$$

This relation between a and b guarantees that we have an ellipse which is tangent to the "cradle" ellipses; we now need to find which of all these possibilities has the smallest area. The area of the ellipse is given by the formula $A = \pi ab$, and so we consider

$$a^2b^2 = 2a^3\big(a - \sqrt{a^2-4}\big)\ .$$

Equating the derivative of the right hand side to zero and solving, we have

$$6a^2\big(a - \sqrt{a^2-4}\big) + 2a^3\Big(1 - \frac{a}{\sqrt{a^2-4}}\Big) = 0\,.$$

Collecting terms gives

$$8a^3 = 6a^2\sqrt{a^2-4} + \frac{2a^4}{\sqrt{a^2-4}}$$

and therefore

$$4a\sqrt{a^2-4} = 3(a^2-4) + a^2 = 4(a^2-3)$$

This implies that $a^2(a^2-4) = (a^2-3)^2$ and so $2a^2 = 9$. Therefore, $a^2 = \frac{9}{2}$, which after simplification gives $b^2 = 6$. So Thomas should choose the ellipse

$$\frac{x^2}{9/2} + \frac{y^2}{6} = 1\ ,$$

which has area $A = \pi ab = 3\pi\sqrt{3}$.

Before we tell Thomas to start constructing the nursery, we'd better check to see whether he might get a better result by taking $q = 0$, which so far we have neglected. This will give $a^2 = 4$, and allowable values for b^2 will be determined by the requirement that the "boundary" ellipse must contain the "cradles": thus, the y value on the boundary ellipse in the first quadrant must be no less than the y value on the cradle. This gives

$$b^2\Bigl(1 - \frac{x^2}{4}\Bigr) \ge 4\bigl(1 - (x-1)^2\bigr)$$

whenever $0 < x < 2$, which simplifies to

$$b^2 \ge 16\Bigl(1 - \frac{2}{2+x}\Bigr)\ .$$

If x approaches 2 then the right hand side continually increases and approaches a value of 8. Therefore the minimum possible value for b^2 is 8, and the minimum area for this ellipse is $4\pi\sqrt{2}$. This is larger than our previous answer, which confirms that the ellipse we found previously is the minimal solution.

NOW TRY Problem 1671.

Q1658 Alexander and Benjamin want to access their favourite computer game. Each has to enter a password, which will be a string of letters a and b. If their words can be converted to each other by substituting aab for bba or vice versa, more than once if necessary, then the game app will agree that the passwords match and will let them access the game. For example, $aaaabab$ and $bbabbba$ match because of the chain of substitutions

$$aa\underline{aab}ab \sim aabb\underline{aab} \sim \underline{aab}bba \sim bbabbba\ .$$

The twins enter their passwords and hit return… nothing happens! They have made a typing error. Even worse, the backspace/delete keys have frozen!! The only hope is to keep typing and see if the passwords match at some time in the future.

(a) Is this ever possible? That is, are there two non–matching passwords which can be extended to give matching passwords?

(b) Suppose that after realising their mistake, Alexander and Benjamin are very careful to type exactly the same in the future. Now is it possible for them to gain access? In other words, are there two non–matching passwords which can be extended *in the same way* to give matching passwords?

SOLUTION For any passwords u, v using letters a, b only, we write $u \sim v$ to denote that u and v are related by a single substitution, and $u \approx v$ to denote that the passwords match (that is, they are related by a chain of substitutions).

For part (a), suppose that the "first attempt" passwords are u and v, and that they are extended by words w_1 and w_2. We want an example for which

$$u \not\approx v \quad \text{but} \quad uw_1 \approx vw_2\ .$$

There are many examples like this. For instance, if $u = aa$ and $v = bb$ then it is clear that u and v do not match, since they are not the same as they stand, and with only two letters, no substitutions are possible. However, appending $w_1 = b$ and $w_2 = a$ respectively gives aab and bba, which are matching passwords.

For part (b) we want

$$u \not\approx v \quad \text{but} \quad uw \approx vw \ ;$$

unfortunately for Alexander and Benjamin, this is not possible. To prove this we shall show that if u, v are any words, x is any single letter and $ux \approx vx$, then $u \approx v$. Starting with $uw \approx vw$ and removing one letter at a time, this will eventually show that $u \approx v$; in other words, the "first attempt" passwords did match after all.

So, suppose that $ux \approx vx$. This means that there is a chain of single substitutions

$$ux \sim u_1x_1 \sim u_2x_2 \sim \cdots \sim u_nx_n \sim vx \ ,$$

where $u_1, u_2, \ldots, u_n$ are words and $x_1, x_2, \ldots, x_n$ are single letters. We consider various cases.

Firstly suppose that all the x_k are equal to x. Since any single substitution changes every letter it involves, none involves the letters x_k; every substitution occurs within the words u_k. But this means that

$$u \sim u_1 \sim u_2 \sim \cdots \sim u_n \sim v$$

and therefore $u \approx v$.

Next, suppose that none of the x_k is equal to x. Since there are only two allowable letters, they must all be equal to the other: call it y. Then we have

$$ux \sim u_1y \sim u_2y \sim \cdots \sim u_ny \sim vx \ .$$

Since a substitution has clearly been made at the end of ux, the last two letters of u must be yy, and the preceding part of the word is unchanged by this substitution. Giving a similar argument involving vx, we can write

$$u = u_0yy \ , \quad u_1 = u_0xx \ , \quad u_n = v_0xx \ , \quad v = v_0yy \ .$$

Now looking at the central part of the above chain,

$$u_1y \sim u_2y \sim \cdots \sim u_ny \ ;$$

therefore by our first case we have $u_1 \approx u_n$, that is,

$$u_0xx \approx v_0xx \ .$$

The words u_0x and v_0x are clearly shorter than u and v, so we may assume by way of induction that our main result applies to these words. Therefore $u_0xx \approx v_0xx$ implies

$u_0x \approx v_0x$, which in turn implies $u_0 \approx v_0$; hence $u_0yy \approx v_0yy$, that is, $u \approx v$, and the proof of this case is finished.

Finally we consider the case when the sequence of letters $x_1, x_2, \ldots, x_n$ may consist of any arrangement of xs and ys. The sequence may then be considered as a number of xs, followed by a number of ys, followed by further xs, and so on; and the result follows from the first two cases.

Q1659 Looking ahead a few years… On their first day at school, Alexander and Benjamin are amazed to find that their class consists entirely of twins! – nine pairs of twins, to be exact. The teacher wants to split the class up for three different activities: 7 of the children will do music, 6 will do reading and 5 will do painting. Each pair of twins will do two different activities. In how many ways can the teacher allocate children to activities?

SOLUTION First we decide which pairs do which activities; we shall allocate individual children later. For each pair, write down the initial of the activity they *will not* be involved in. For example, If Alexander and Benjamin do music and painting, Christine and Denise do music and reading and so on, we write $RP\cdots$. There will be two pairs of twins who don't do music, three who don't do reading and four who don't do painting; so the number of ways to allocate activities is the same as the number of ways to write down a nine–letter word consisting of two Ms, three Rs and four Ps. The number of ways to arrange 9 letters is $9!$. However, interchanging the two Ms will give the same word, and so we must divide by $2!$; similarly, we must divide by $3!$ on account of the repeated Rs and by $4!$ on account of the repeated Ps. The number of words is therefore

$$\frac{9!}{2!\,3!\,4!} = 1260\ .$$

To complete the allocation, we must decide which of each pair of twins does which activity. There are 2 choices for each pair, 2^9 altogether. Therefore the total number of options available to the teacher is

$$\frac{9!}{2!\,3!\,4!}\,2^9 = 1260 \times 512 = 645120\ .$$

NOW TRY Problem 1672.

Q1660 Mindful of Alexander and Benjamin's future mathematical education, Thomas assigns a quadratic to each of them:

$$a(n) = 21n^2 + 26n + 8 \quad\text{and}\quad b(n) = 10n^2 + 11n + 3\ .$$

Some time in the future, with careful study and hard work, the twins will be able to show that for any value of the integer n, their expressions $a(n)$ and $b(n)$ will never have any common factor greater than 1. Can you do it now?

SOLUTION First note that we have the factorisations

$$a(n) = (3n + 2)(7n + 4) \quad\text{and}\quad b(n) = (2n + 1)(5n + 3)\ .$$

Now if $a(n)$ and $b(n)$ have a common factor greater than 1, they must have a common prime factor p. Using the above factorisations, we have

$$p \mid 3n+2 \text{ or } p \mid 7n+4 \; ; \quad \text{and} \quad p \mid 2n+1 \text{ or } p \mid 5n+3 \; .$$

If p is a factor of $3n+2$ and of $2n+1$, then it is a factor of

$$2(3n+2) - 3(2n+1) = 1 \; ,$$

which is impossible; and the other three possibilities are eliminated by similar arguments. Therefore $a(n)$ and $b(n)$ have no common factor greater than 1.