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An inequality inspired by Haruki's Lemma

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1 Introduction

It is always interesting to look at known and established facts from a different perspective. Euclidean geometry is rich with beautiful classical theorems. In some cases, it is possible to interpret them in the context of inequalities. The obtained geometric inequalities help us to understand plane geometry from a different angle. This article is an attempt to consider a popular result from plane geometry known as Haruki's Lemma [3, 4] in an unorthodox way.

We start by explaining Haruki's Lemma. Denote the intersection point of sides AB and AC of a triangle ABC and a chord GH of its circumcircle by N and I, respectively; see Figure 1. The Lemma says that, when point A moves along the arc GH of the fixed circle for fixed points B, C, G and H, the ratio $\frac{|GN||IH|}{|NI|}$ does not change. This lemma has some interesting applications including an elegant proof of The Butterfly Theorem[2] [1].

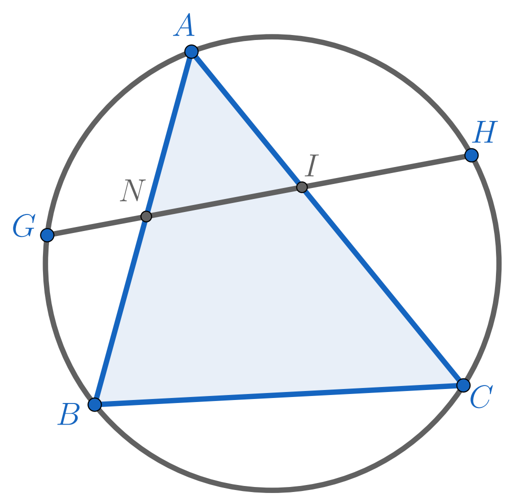

Figure 1: Haruki's Lemma

All figures in this paper were created using the website www.geogebra.org [6].

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[2] For more information on the Butterfly Theorem, and for a beautiful generalisation of that theorem, see the article *The Multi-Butterfly Theorem* by Martina Skorpilova in this issue of *Parabola*.

2 Main Result

The following theorem gives a different look at Haruki's Lemma through the glasses of inequalities.

Theorem 1.
Suppose that chord GH of a circumscribed circle of triangle ABC intersects sides AB and AC so that points G and C are on different arcs AB of the circle and that points H and B are on different arcs AC of the circle; see Figure 2. Let I be the intersection point of AC and GH and let M be the intersection point of the line passing through G and H and the circle passing through A, B and I. Then

$$|MH| \sin \angle A \leq |CG| \,.$$

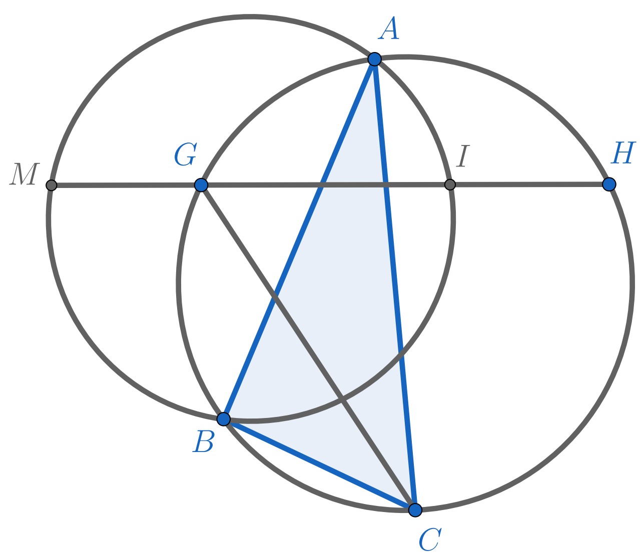

Figure 2: Theorem 1

Proof. We will use the fact that, if points B, C, G, H are fixed on the fixed circumscribed circle of the triangle ABC and point A moves along the arc GH, then $\angle A$ does not change and therefore $\angle BMH$ does not change either. Since line GH is fixed, the position of point M also doesn't change. Therefore, during this movement of point A, none of the quantities in the inequality $|MH| \sin \angle A \leq |CG|$ change. Consequently, it is sufficient to prove the inequality in one special case. We will let A approach G and prove the inequality in this limiting case. Now, points A, G, I coincide and CA is a tangent to the circle passing through B, I and A; see Figure 3.

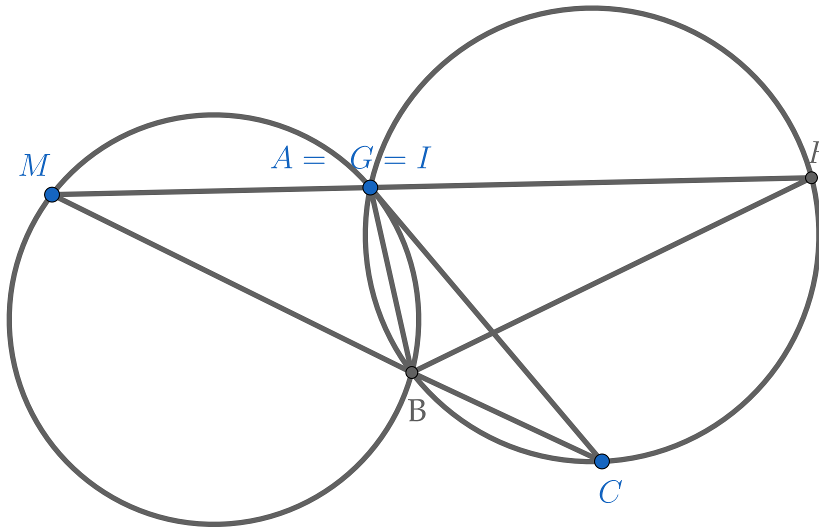

Figure 3: A limiting case

We can rewrite the inequality as $|MH| \leq \frac{|CG|}{\sin \angle A}$. It is sufficient to prove that this inequality holds when MH is maximal. This happens when MH is perpendicular to AB. Indeed, when MH rotates around point A, the angles MBH, BMH and MHB do not change. So, the triangle MBH remains similar to the original triangle MBH. One of its sides MH is largest when its other sides are largest. But MB and BH are largest when they are diameters of the corresponding circles. This happens when $\angle MAB = 90°$. Let us denote these new positions of M and H by M' and H'; see Figure 4.

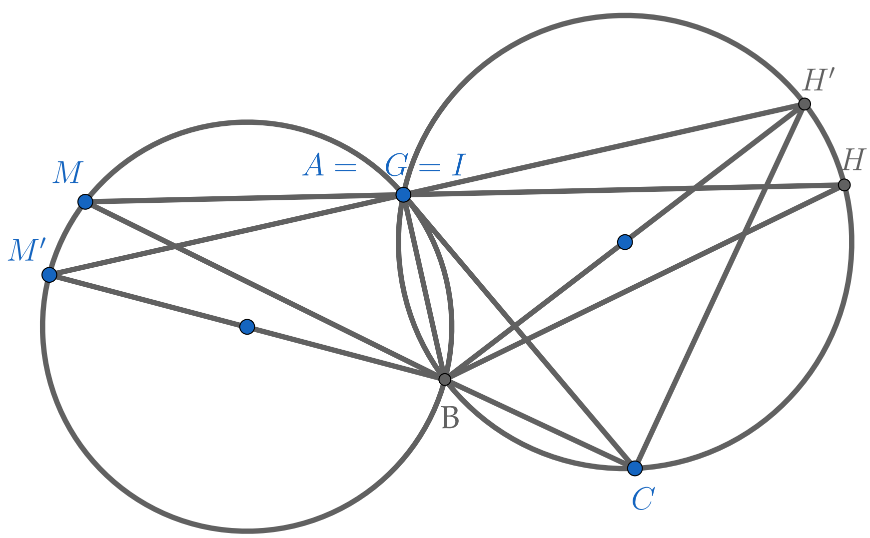

Figure 4: New positions M' and H'

So, we only need to prove that $|M'H'| \leq \frac{|CA|}{\sin \angle BAC}$. We will actually prove that there is equality; that is, that $|M'H'| = \frac{|CA|}{\sin \angle BAC}$. First, note that $\angle BAC = \angle BM'A$ and $\angle ACB = \angle M'H'B$. The triangles ACB and $M'H'B$ are therefore similar. This means that we can write $\frac{|M'H'|}{|AC|} = \frac{|BH'|}{|BC|}$. But BH' is a diameter; therefore, $\frac{|BC|}{|BH'|} = \sin \angle CH'B$. Since $\angle CH'B = \angle CAB$, we obtain $\frac{|M'H'|}{|AC|} = \frac{1}{\sin \angle CAB}$, which gives us the necessary equality $|M'H'| \sin \angle BAC = |CA|$. This completes the proof. $\square$

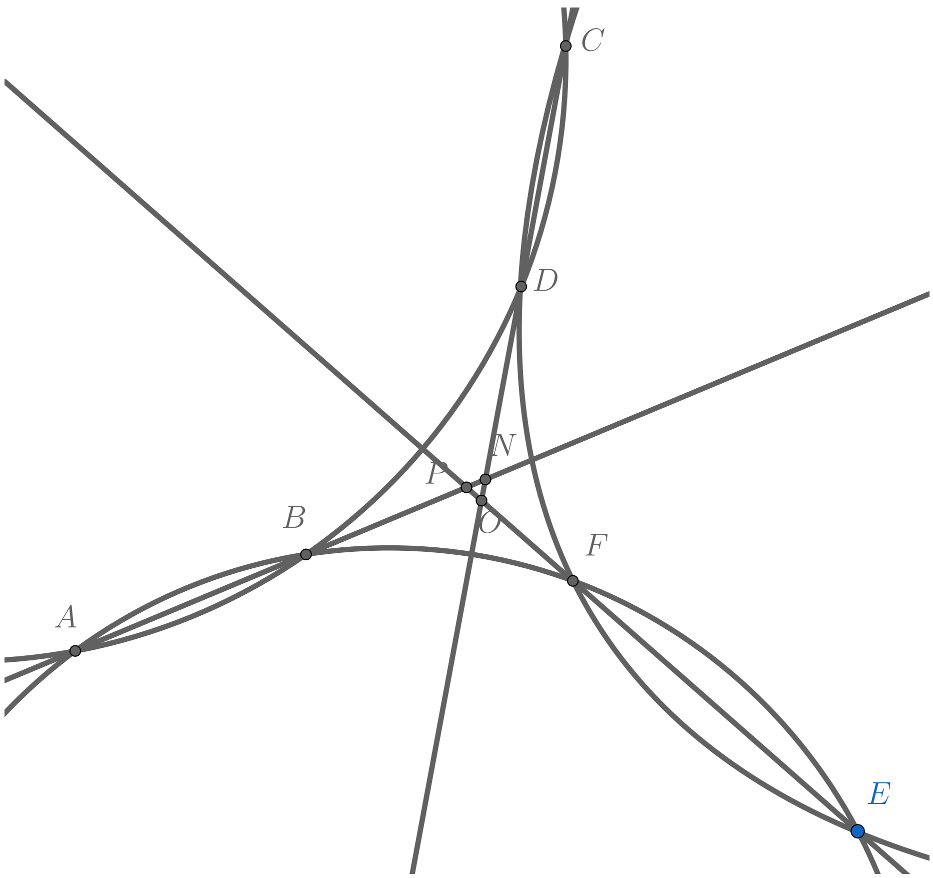

Figure 5: Haruki's Theorem

3 Remarks

This paper would not be complete without mention of another result of Hiroshi Haruki, known as Haruki's Theorem [2]. Suppose that three circles intersect at points A, B, C, D, E and F as in Figure 5. The Theorem says that

$$\frac{|AD|}{|DE|} \frac{|EB|}{|BC|} \frac{|CF|}{|FA|} = 1 \, .$$

The key observation for the proof of this theorem is the fact that AB, CD and EF are concurrent. Surprising enough, this can also be proved using inequalities. Suppose on the contrary that these lines are not concurrent. Then these lines form a triangle as in Figure 5. Let us denote the vertices of this triangle as N, O and P. The other possible orientation of the points N, O and P can be studied in an analogous way. We obtain

$$|NB||NA| > |PB||PA| = |PF||PE| > |OF||OE| = |OD||OC| > |ND||NC = |NB||NA|,$$

which is a contradiction. So, AB, CD and EF are indeed concurrent.

References

[1] A. Bogomolny, *Hiroshi Haruki's Lemma*, https://www.cut-the-knot.org/Curriculum/Geometry/Haruki.shtml, last accessed on 2022-03-14.

[2] H. Haruki, On a metrical theorem in geometry of circles, *Colloq. Math.* **25** (1972), 99–102.

[3] R. Honsberger, Mathematical gems: The Butterfly Problem and other delicacies from the noble art of Euclidean geometry – Part 1, *The Two-Year College Mathematics Journal* **14** (1983), 2–8.

[4] R. Honsberger, *Mathematical Diamonds*, The Mathematical Association of America, 2003.

[5] Y. Bezverkhnyev, Haruki's Lemma and a related locus problem, *Forum Geometricorum* **8** (2008), 63–72.

[6] GeoGebra, https://www.geogebra.org/, last accessed on 2022-03-14.