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# Problems 1671–1680

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**Q1671** As in Problem 1643 and Problem 1663, a Geezer number is a $k$–digit positive integer $d_0 d_1 \cdots d_{k-1}$ in base $10$ which consists of $d_0$ zeros, $d_1$ ones and so on. We have already proved that in a $k$–digit Geezer number the sum of the digits is $k$; and the digits $d_3, d_4, \ldots, d_{k-1}$ include at most one $1$, with all the rest of these digits being $0$. Find all Geezer numbers.

**Q1672** A bag contains $2n$ balls, two each of $n$ different colours. They are to be drawn from the bag in a random order and placed in a row. Prove *without calculation* that the probability of obtaining a row consisting of pairs of the same colour is the same as the probability of obtaining a row in which the second half is the same as the first.

**Q1673** The article by Jonathan Hoseana and Handi Koswara in *Parabola* Volume 57, Issue 2 may help with the following problem.

(a) Find a constant $\alpha$ such that the function

$$\alpha x - \sum_{i=1}^{n} 2^i \sqrt{x^2 + \frac{x}{4^i}} \qquad (*)$$

has a limit as $x \to \infty$.

(b) Suppose that $\alpha$ is the value in (a) and $L_n$ is the corresponding limit (in terms of $n$). Find the limit of $L_n$ as $n \to \infty$.

**Q1674** A sequence of numbers $x_1, x_2, x_3, \ldots$ is generated as follows. We begin with $x_1 = 1$; then we take the cosine and sine of $x_1$; then the cosine and sine of $x_2$; and so on. To clarify:

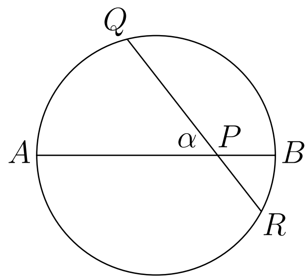
$$\begin{aligned}
x_1 &= 1\\
x_2 &= \cos(x_1) = \cos(1)\\
x_3 &= \sin(x_1) = \sin(1)\\
x_4 &= \cos(x_2) = \cos(\cos(1))\\
x_5 &= \sin(x_2) = \sin(\cos(1))\\
x_6 &= \cos(x_3) = \cos(\sin(1))\\
x_7 &= \sin(x_3) = \sin(\sin(1))\\
x_8 &= \cos(x_4) = \cos(\cos(\cos(1)))
\end{aligned}$$

and so on. Find the smallest $n > 1$ such that $x_n > 0.99$.

**Q1675** Prove that if $a, b, c$ are positive numbers, then

$$\frac{4a+3b+2c}{4c+3b+2a}+\frac{4b+3c+2a}{4a+3c+2b}+\frac{4c+3a+2b}{4b+3a+2c}\geq 3\ .$$

**Q1676** Let $C$ be a circle on diameter $AB$, and let $\alpha$ be an acute angle.




For any point $P$ on $AB$, draw a chord $QR$ of the circle passing through $P$, with the angle between $AB$ and $QR$ being equal to $\alpha$. Given that the quantity

$$PQ^2 + PR^2$$

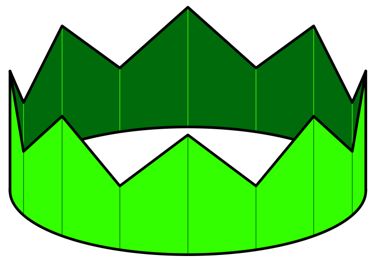
is the same for all choices of $P$, determine the angle $\alpha$; and then find $PQ^2 + PR^2$ in terms of the radius of $C$.

**Q1677** For this question, a *knockout contest* among $n$ entrants means the following. Let $k$ be the integer for which $2^{k-1} < n \leq 2^k$. Then $2^k - n$ players (chosen at random) are "given a bye" in the first round: that is, they progress to the next round without playing a match. The remaining $2n - 2^k$ players play in pairs (once again chosen at random), and the $n - 2^{k-1}$ winners also progress. This leaves $2^{k-1}$ entrants who will play in pairs, leaving $2^{k-2}$ winners in the next round; and so on; until the overall winner is decided by a match between the last $2$ players. Note that there are no byes after the first round, and so the eventual winner will play either $k-1$ or $k$ matches, depending on whether they do or do not receive a bye in the first round.

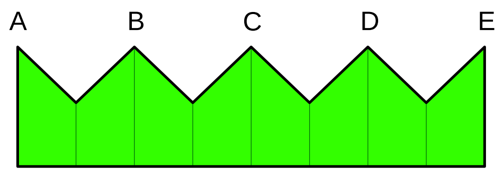
(a) If there are $n$ players at the beginning of the competition, then how many matches will be played altogether?
**Comment** This is a well known problem and there is a very easy solution.
Do not try to consider the number of byes, the number of rounds or other details!

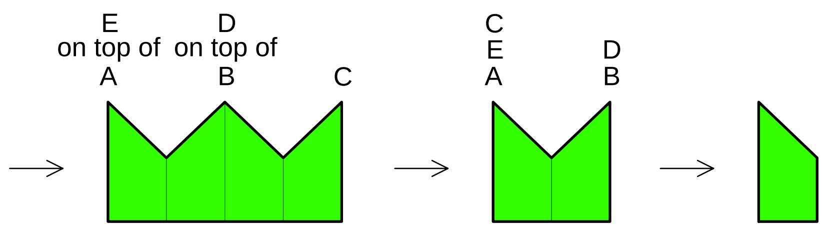
(b) A football club wishes to rank the three strongest arm–wrestlers from a pool of $100$ candidates by using a knockout contest to determine the best arm–wrestler; then another knockout contest to determine the second–best; then another to determine the third–best. Show that if the contests are carefully organised, then this can be accomplished in $113$ matches altogether; but that fewer than $105$ matches will never be enough.

**Q1678** Last Christmas, I pulled a Christmas cracker, and out popped the traditional paper crown.



While inside the cracker it had been flattened out between two opposite points $A$ and $E$, and then folded right half over left three times, as in the diagrams.




It's clear that if the crown is unfolded, then some of the creases that have been made will point towards the outside of the crown, and some will point towards the inside. Is it possible to now refold the crown in the same way as before, but starting with a *different* pair of opposite points instead of $A$ and $E$, and *without* reversing any of the folds already made?

**Q1679** David is designing a tiling pattern for his rectangular bathroom floor. Most people have tilings which consist of rectangles or hexagons, but David thinks this is boring, so he has decided to use pentagons. Any sorts of pentagonal shapes are acceptable: they do not have to be regular pentagons, and they need not all be congruent. Moreover, David has decided that an attractive design should have three further features:

- each corner of the rectangle should belong to only one pentagon, and no two corners can belong to the same pentagon;
- each boundary point of the rectangle should belong to at most two pentagons;
- each interior point of the rectangle which is a corner of a pentagon should belong to exactly two other pentagons.

Show that in order to fulfil these requirements, David must use exactly $8$ pentagons; and draw a possible design for the bathroom floor.

**Q1680** Calculate the limit of the sum

$$S_n = \sum_{k=0}^{n-1} \frac{n}{(n+k)^2}$$

as $n$ tends to $\infty$.