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Inequalities related to foci of hyperbola

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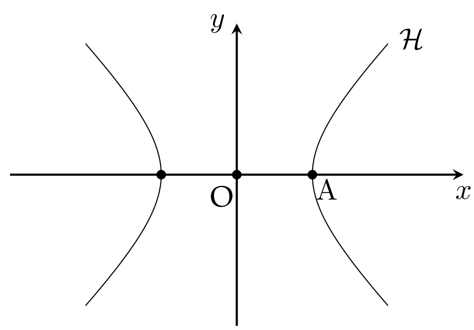
1 Introduction

We consider the hyperbola $\mathcal{H}$ whose points (x, y) are determined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

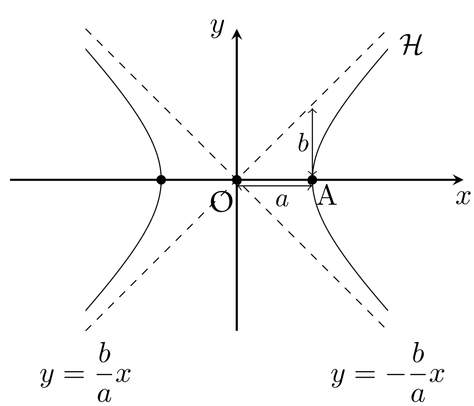
with $a, b > 0$ constant:



Since $(a, 0)$ is a solution to Equation (1), it is a point of $\mathcal{H}$ and so a represents the distance between the origin O and each intersection point A between $\mathcal{H}$ and the x-axis. On the other hand, we can say that b represents how open $\mathcal{H}$ is. This is because the asymptotes of $\mathcal{H}$ are the lines given by

$$y = \pm\frac{b}{a}x\,,$$

as follows:



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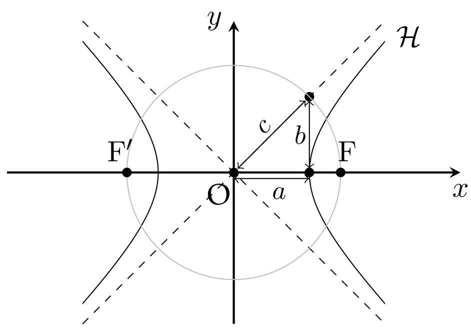
In other words, $\mathcal{H}$ opens and closes while approaching the asymptote as b varies, and so b appears *implicitly* in the graph of $\mathcal{H}$ as opposed to a which appears explicitly.

The hyperbola $\mathcal{H}$ has two so-called *focal points*, or *foci*; these are the points F and F' given by

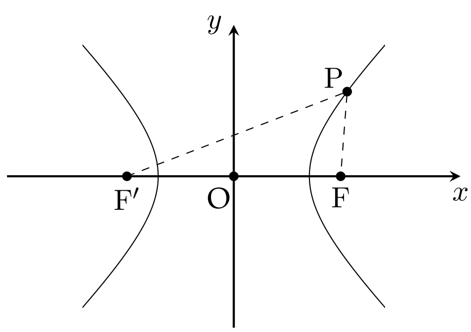
$$\big(\sqrt{a^2+b^2},0\big) \quad \text{and} \quad \big(-\sqrt{a^2+b^2},0\big)\,, \tag{2}$$

respectively or, in other words, $(\pm c, 0)$ where $c = \sqrt{a^2+b^2}$:



The foci F and F' are determined by the values of a and b and thereby by the hyperbola $\mathcal{H}$. However, the hyperbola (and thereby also a and b) is also determined by the foci; in particular, the hyperbola $\mathcal{H}$ consists exactly of all points P whose distances to F and to F' are the same except for a difference of $2a$. That is,

$$\big||\mathrm{PF}| - |\mathrm{PF}'|\big| = 2a\,. \tag{3}$$



In fact, it turns out that the hyperbola $\mathcal{H}$ consists exactly of all points $P = (x, y)$ that satisfy Equation (1). In other words, Equations (1) and (3) are two equivalent ways of expressing the points in $\mathcal{H}$.

To see that Equation (1) follows from Equation (3), note that

$$\pm 2a = |\mathrm{PF}| - |\mathrm{PF}'| = \sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2}\,;$$

i.e.,

$$\sqrt{(x-c)^2+y^2} = \sqrt{(x+c)^2+y^2} \pm 2a\,.$$

Squaring both sides of this equation gives

$$(x-c)^2 + y^2 = (x+c)^2 + y^2 \pm 4a\sqrt{(x+c)^2+y^2} + 4a^2,$$

i.e.,

$$\mp a\sqrt{(x+c)^2+y^2} = cx + a^2\,.$$

By considering the squares of both sides of this equation, we have

$$(c^2-a^2)x^2 - a^2y^2 = (c^2-a^2)a^2\,. \tag{4}$$

We can set $b^2 := c^2 - a^2$ since $c - a > 0$ and can represent Equation (4) as

$$b^2x^2 - a^2y^2 = a^2b^2\,.$$

Hence, we obtain Equation (1) by dividing both sides by a^2b^2.

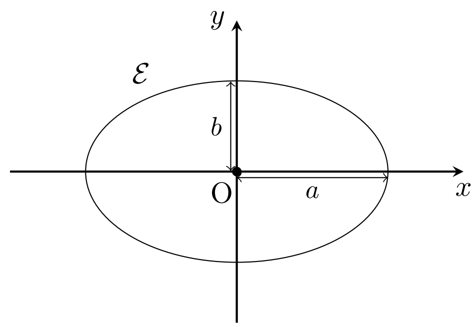
Reverse calculations show conversely that Equation (1) implies Equation (3).

2 Hyperbolas and ellipses

Before proceeding to the main part of the paper, it is interesting to compare hyperbolas and ellipses. Changing a sign in Equation (1) for hyperbolas $\mathcal{H}$ gives the equation

$$\mathcal{E}:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{5}$$

The points $P = (x, y)$ satisfying this equation form the ellipse with a and b representing its largest and smallest radius, respectively, if it is assumed that $a \geq b$:



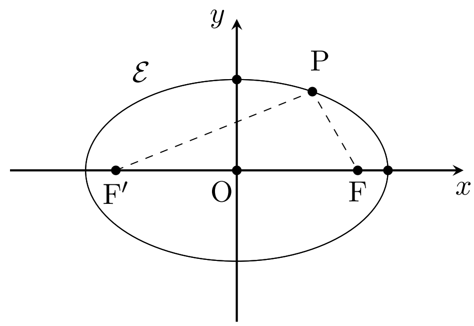
Like the hyperbola $\mathcal{H}$, the ellipse $\mathcal{E}$ has foci F and F′, the same but for a change of sign:

$$\left(\sqrt{a^2-b^2}, 0\right) \quad \text{and} \quad \left(-\sqrt{a^2-b^2}, 0\right), \tag{6}$$

respectively or, in other words, $(\pm c, 0)$ where $c = \sqrt{a^2-b^2}$. Just like for the hyperbola, these foci also define the set of points in the ellipse $\mathcal{E}$. In particular, the points P in the ellipse are those whose sum of distances to F and to F′ equal $2a$:

$$|\mathrm{PF}| + |\mathrm{PF}'| = 2a\,. \tag{7}$$

Note that this equation is obtained by changing a sign in Equation (3).



Quadratic curves of the hyperbola, ellipse, etc. have many other properties; see textbooks on quadratic curves and analytic geometry, e.g. [2, 3], for more details.

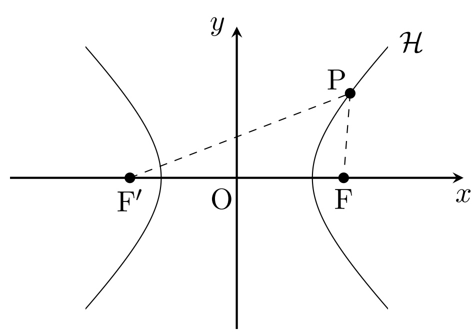
3 Main results and their proofs

The main result of this paper is a new inequality, given by Theorem 2 below. To prove this result, another inequality is required.

Proposition 1. *Let* P *be any point on* $\mathcal{H}$ *and let* F *and* F′ *be the foci of* $\mathcal{H}$*. Then*

$$|\mathrm{PF}| + |\mathrm{PF}'| \geq 2\sqrt{a^2 + b^2}\,, \tag{8}$$

with equality exactly when P *lies on the* x*-axis.*



Proof. By the Triangle Inequality, applied to the triangle $\triangle\mathrm{PFF}'$, and Equation (2),

$$|\mathrm{PF}| + |\mathrm{PF}'| \geq \overline{\mathrm{FF}'} = 2\sqrt{a^2 + b^2}\,.$$

Equality occurs exactly when triangle $\triangle\mathrm{PFF}'$ is a line; i.e., when P is on the x-axis. □

Theorem 2. *Let* P *be any point on* $\mathcal{H}$ *and let* F *and* F′ *be the foci of* $\mathcal{H}$*. Then*

$$|\mathrm{PF}||\mathrm{PF}'| \geq b^2\,, \tag{9}$$

with equality exactly when P *lies on the* x*-axis.*

Proof. By definition of hyperbolas, in particular Equation (3),

$$\begin{aligned}
|\mathrm{PF}||\mathrm{PF}'| &= |\mathrm{PF}||\mathrm{PF}'| + \frac{1}{4}(2a)^2 - a^2 \\
&= |\mathrm{PF}||\mathrm{PF}'| + \frac{1}{4}\big(|\mathrm{PF}| - |\mathrm{PF}'|\big)^2 - a^2 \\
&= \frac{1}{4}\big(|\mathrm{PF}|^2 + |\mathrm{PF}'|^2 + 2|\mathrm{PF}||\mathrm{PF}'|\big) - a^2 \\
&= \frac{1}{4}\big(|\mathrm{PF}| + |\mathrm{PF}'|\big)^2 - a^2\,.
\end{aligned}$$

By Proposition 1,

$$|\mathrm{PF}||\mathrm{PF}'| \geq \big(a^2 + b^2\big) - a^2 = b^2\,,$$

with equality exactly when P lies on the x-axis. □

It is interesting to note that Inequality (9) depends only on the parameter b and not on the parameter a.

References

[1] N. Someyama and M.L.A. Borongan, New proofs of triangle inequalities, *Journal of Ramanujan Society of Mathematics and Mathematical Sciences* **7** (Dedicated to Prof. A.K. Agarwal's 70th Birth Anniversary, 2019), 85–92.

[2] G.B. Thomas and R.L. Finney, *Calculus and Analytic Geometry* (Ninth Edition), Addison Wesley, 1998.

[3] K. Yano, *Analytic Geometry* (in Japanese), Asakura Publishing, 2004.