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The Multi-Butterfly Theorem

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1 Introduction

The *Butterfly Theorem* is an elegant result on chords of a circle. Its history dates back to the early nineteenth century. A discovered family archive of William Wallace (1768–1843) shows that the theorem was proven by this Scottish mathematician, astronomer and inventor in 1805; see [2]. Moreover, William Wallace also dealt with a more general statement for conics two years earlier.

According to [1], the appellation of the theorem was first publicised in [6] in 1944.

Various proofs as well as miscellaneous generalizations of the theorem have been gradually published; see, for example, [1, 3, 5, 7]. We present a generalization that is valid for n circles. It has already been proven for $n = 2$, the so-called *Better Butterfly Theorem*, by Qiu Fawen and his students in 1997; see [4].

If we talk about chords, then we suppose that they are mutually distinct. We denote the distance between two points A and B by $|AB|$ and the size of the angle ABC by $|\angle ABC|$.

2 The Butterfly Theorem

Now, we formulate the *Butterfly Theorem*.

Theorem 1 (The Butterfly Theorem)**.**
Let k be a given circle, and let S be the midpoint of its any chord XY (see Figure 1). Let AB, CD be two chords of k passing through S (points A, D lie in the same half-plane with edge XY). If U, V are the intersections of the chord XY with the chords AC, BD, respectively, then S is also the midpoint of UV.

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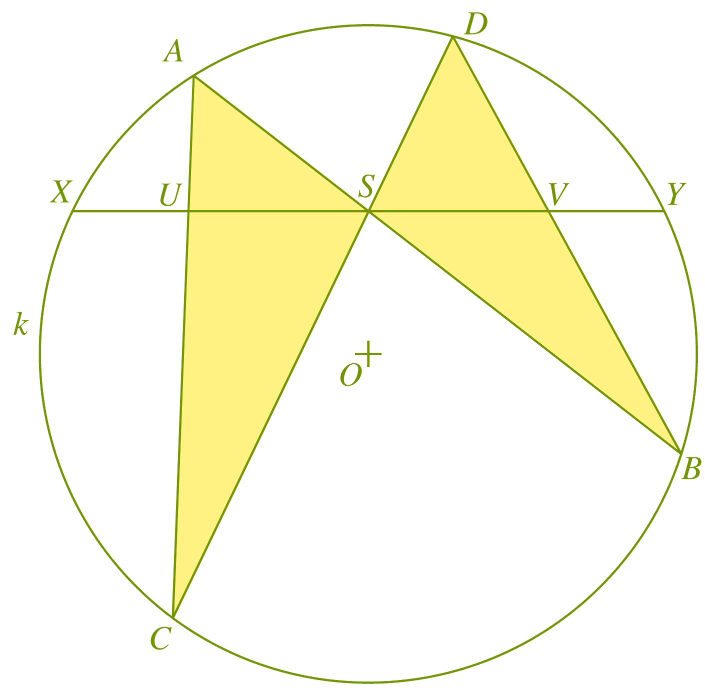

Figure 1.

3 The Multi-Butterfly Theorem

We now present the generalization of the theorem for n circles.

Theorem 2 (The Multi-Butterfly Theorem)**.**
Let $k_1, k_2, \ldots, k_n$ be concentric circles with the common center O (see Figure 2 for $n = 4$). Assume, without loss of generality, that the inequality $r_i \leq r_{i+1}$ holds for radii r_i of k_i, $i = 1, 2, \ldots, n-1$. Let XY be any chord of k_n, and let S be its midpoint. Let A_nB_n, C_nD_n be chords of k_n which pass through S (points A_n, D_n lie in the same half-plane with edge XY). Let $U_1, U_2, \ldots, U_{n-1}, U_n$ be the intersections of the chord XY with the chords A_1C_2, A_2C_3, $\ldots$, $A_{n-1}C_n$, A_nC_1, respectively, and, further, let V_1, V_2, $\ldots$, V_{n-1}, V_n be the intersections of the chord XY with the chords D_1B_2, D_2B_3, $\ldots$, $D_{n-1}B_n$, D_nB_1. Then the following relation holds:

$$\frac{1}{|SU_1|} + \frac{1}{|SU_2|} + \cdots + \frac{1}{|SU_{n-1}|} + \frac{1}{|SU_n|} = \frac{1}{|SV_1|} + \frac{1}{|SV_2|} + \cdots + \frac{1}{|SV_{n-1}|} + \frac{1}{|SV_n|}\,.$$

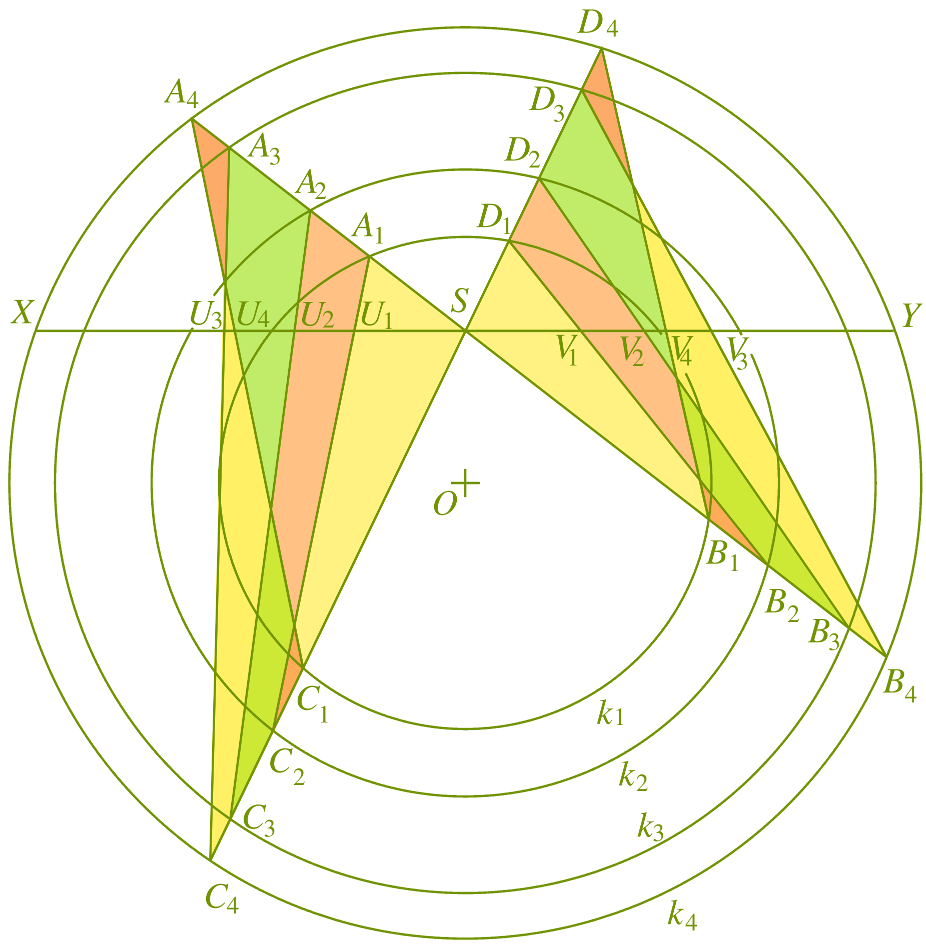

Figure 2.

Proof. We verify our statement by mathematical induction. Firstly, we show that the theorem holds for $n = 1$; i.e., we prove the equality

$$\frac{1}{|SU_1|} = \frac{1}{|SV_1|}\,. \tag{1}$$

The equality (1) is clearly equivalent to $|SU_1| = |SV_1|$; i.e., the *Butterfly Theorem*. As stated above, diverse proofs of this theorem have been published. We present a method that is inspired by the proof of the already mentioned *Better Butterfly Theorem*; see [4].

Now, we denote the sizes of angles as shown in Figure 3:

$$|\angle U_1SA_1| = |\angle V_1SB_1| = \alpha\,,$$
$$|\angle U_1SC_1| = |\angle V_1SD_1| = \beta\,.$$

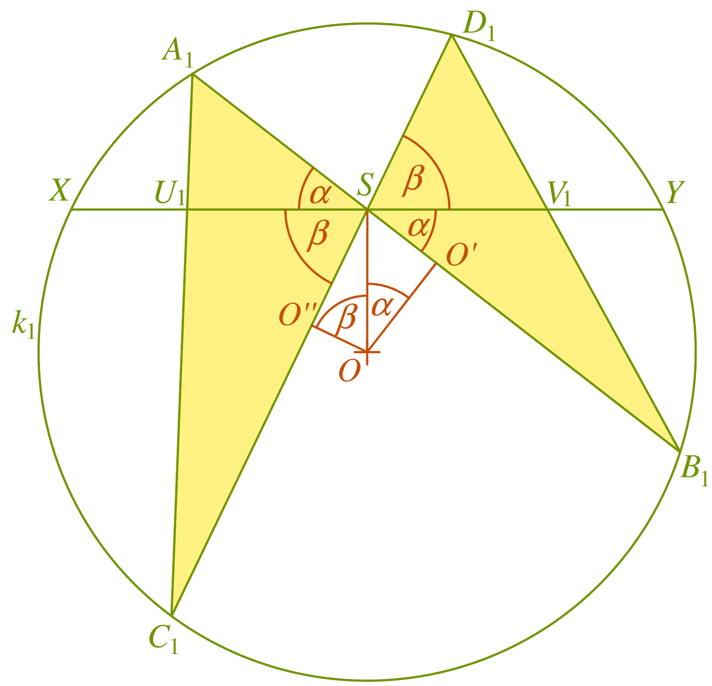

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Figure 3.

The relation (1) is equivalent to

$$\frac{\sin(\alpha+\beta)}{|SU_1|} = \frac{\sin(\alpha+\beta)}{|SV_1|}\,. \tag{2}$$

Further, let us have a look at the triangle SA_1C_1. Since its area is the sum of areas of triangles SA_1U_1 and SC_1U_1, we get

$$\frac{1}{2}\,|SA_1|\,|SC_1|\sin(\alpha+\beta) = \frac{1}{2}\,|SA_1|\,|SU_1|\sin\alpha + \frac{1}{2}\,|SC_1|\,|SU_1|\sin\beta\,.$$

By multiplying this equality by the expression $\frac{2}{|SA_1||SC_1||SU_1|}$, we obtain

$$\frac{\sin(\alpha+\beta)}{|SU_1|} = \frac{\sin\alpha}{|SC_1|} + \frac{\sin\beta}{|SA_1|}\,. \tag{3}$$

Analogously, using triangle SD_1B_1, we have

$$\frac{\sin(\alpha+\beta)}{|SV_1|} = \frac{\sin\alpha}{|SD_1|} + \frac{\sin\beta}{|SB_1|}\,.$$

The relation (2) is therefore valid if and only if

$$\frac{\sin\alpha}{|SC_1|} + \frac{\sin\beta}{|SA_1|} = \frac{\sin\alpha}{|SD_1|} + \frac{\sin\beta}{|SB_1|}\,. \tag{4}$$

If the chord XY passes through the point O, then $|SA_1| = |SB_1| = |SC_1| = |SD_1|$. Hence, (4) holds, which proves the *Multi-Butterfly Theorem* for $n = 1$.

If the chord XY does not pass through the point O, then we rearrange (4) as follows, assuming without loss of generality that $|SA_1| < |SB_1|$ and, thus, that $|SD_1| < |SC_1|$:

$$\left(\frac{1}{|SA_1|} - \frac{1}{|SB_1|}\right) \sin\beta = \left(\frac{1}{|SD_1|} - \frac{1}{|SC_1|}\right) \sin\alpha\,. \tag{5}$$

Using simple algebra, we obtain the equivalent equality

$$\frac{|SB_1| - |SA_1|}{|SA_1|\,|SB_1|} \sin\beta = \frac{|SC_1| - |SD_1|}{|SD_1|\,|SC_1|} \sin\alpha\,. \tag{6}$$

Since inscribed angles subtended by the same arc are equal, we have by Figure 3 that $|\angle C_1A_1B_1| = |\angle C_1D_1B_1|$ and $|\angle A_1C_1D_1| = |\angle A_1B_1D_1|$. The triangles A_1C_1S and D_1B_1S are therefore similar. It follows that

$$\frac{|SA_1|}{|SD_1|} = \frac{|SC_1|}{|SB_1|}$$

or, equivalently,

$$|SA_1|\,|SB_1| = |SC_1|\,|SD_1|\;. \tag{7}$$

We just proved the so-called *Intersecting Chords Theorem*: If two chords A_1B_1 and C_1D_1 of a circle intersect in a point S, then the equality $|SA_1|\,|SB_1| = |SC_1|\,|SD_1|$ holds.

Let O' and O'' denote the feet of the perpendiculars from the center O of k_1 to the chords A_1B_1 and C_1D_1, respectively. Then, $\alpha = |\angle SOO'|$ and $\beta = |\angle SOO''|$. Since the points O' and O'' are the midpoints of the chords A_1B_1 and C_1D_1, we see that $|SB_1| - |SA_1| = 2\,|SO'|$ and $|SC_1| - |SD_1| = 2\,|SO''|$. Thus,

$$\begin{aligned} |SB_1| - |SA_1| &= 2\,|SO| \sin\alpha\,, \\ |SC_1| - |SD_1| &= 2\,|SO| \sin\beta\,. \end{aligned} \tag{8}$$

After substituting (7) and (8) into (6), and after subsequent re-arrangements, we obtain

$$\frac{2\,|SO| \sin\alpha}{|SC_1|\,|SD_1|} \sin\beta = \frac{2\,|SO| \sin\beta}{|SD_1|\,|SC_1|} \sin\alpha\,.$$

This equality, which re-states (6), is clearly true. Since (6) is equivalent to (1), we have proved the *Multi-Butterfly Theorem* for $n = 1$, i.e., the *Butterfly Theorem*.

Now, we assume that the statement holds for $n = j$, and we prove that it holds also for $n = j + 1$; see Figure 4.

Thus, we assume that the equality

$$\frac{1}{|SU_1|} + \frac{1}{|SU_2|} + \cdots + \frac{1}{|SU_j|} = \frac{1}{|SV_1|} + \frac{1}{|SV_2|} + \cdots + \frac{1}{|SV_j|} \tag{9}$$

holds for j circles. We want to show that (9) implies the identity

$$\frac{1}{|SU_1|} + \frac{1}{|SU_2|} + \cdots + \frac{1}{|S\bar{U}_j|} + \frac{1}{|S\bar{U}_{j+1}|} = \frac{1}{|SV_1|} + \frac{1}{|SV_2|} + \cdots + \frac{1}{|S\bar{V}_j|} + \frac{1}{|S\bar{V}_{j+1}|} \tag{10}$$

for $j+1$ circles.

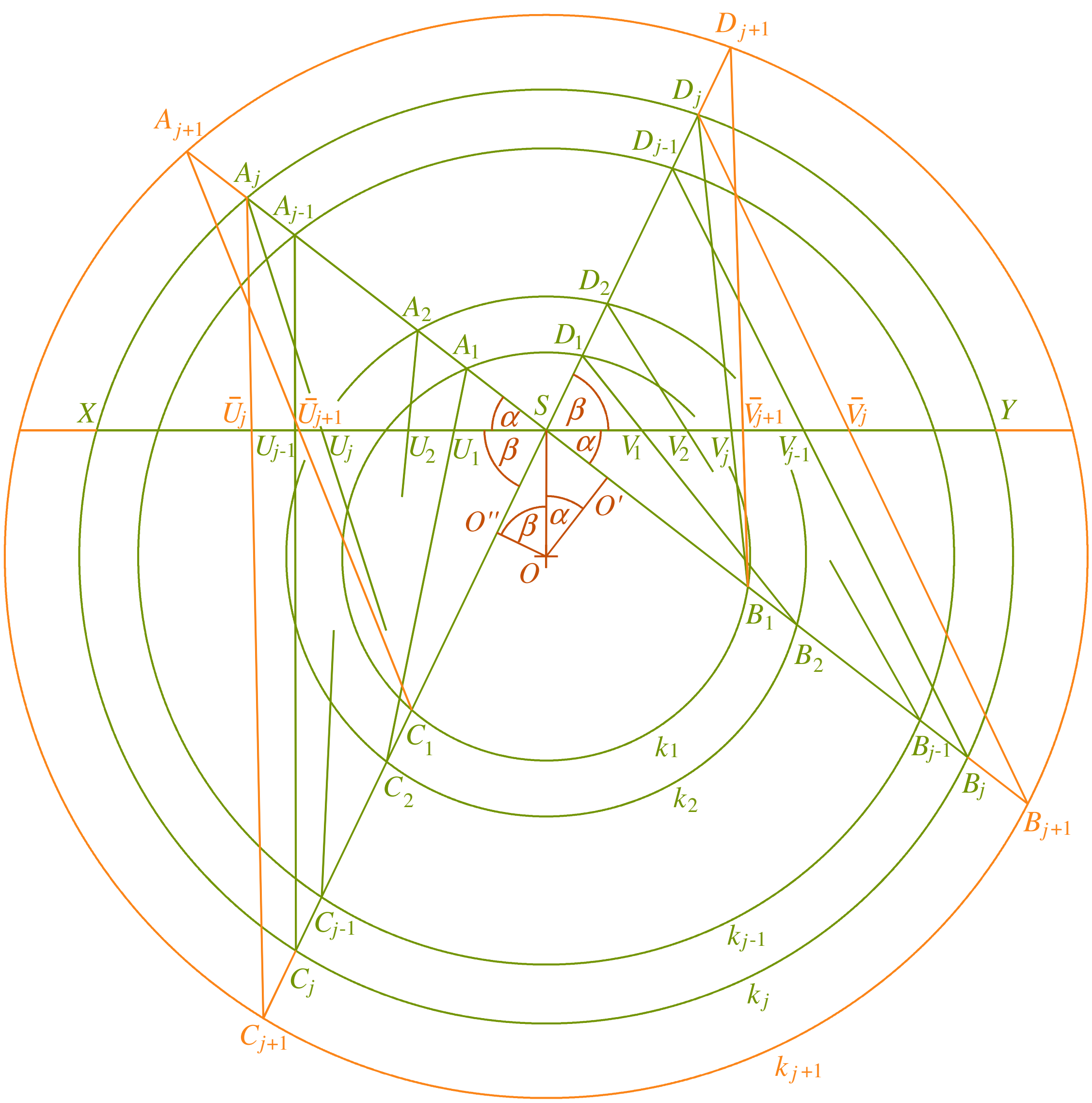

Figure 4.

Note that there are the points U_j, V_j in (9), while there are the points $\bar{U}_j$, $\bar{V}_j$ in (10). For $n = j+1$, we have the chord A_jC_{j+1} instead of the chord A_jC_1, and, analogously, we have the chord D_jB_{j+1} instead of the chord D_jB_1. We denote the intersections of the chord XY with the chords A_jC_{j+1}, D_jB_{j+1}, $A_{j+1}C_1$, $D_{j+1}B_1$ by $\bar{U}_j$, $\bar{V}_j$, $\bar{U}_{j+1}$, $\bar{V}_{j+1}$, respectively.

Further, if we define

$$G = \frac{1}{|SU_1|} + \frac{1}{|SU_2|} + \cdots + \frac{1}{|SU_{j-1}|}\,,$$
$$H = \frac{1}{|SV_1|} + \frac{1}{|SV_2|} + \cdots + \frac{1}{|SV_{j-1}|}\,,$$

then we can rewrite (9) and (10) as

$$G - H = \frac{1}{|SV_j|} - \frac{1}{|SU_j|}\text{(induction hypothesis)}$$

and
$$G - H = \frac{1}{|S\bar{V}_j|} + \frac{1}{|S\bar{V}_{j+1}|} - \frac{1}{|S\bar{U}_j|} - \frac{1}{|S\bar{U}_{j+1}|}\,.$$

Thus, it suffices to show that

$$\frac{1}{|SV_j|} - \frac{1}{|SU_j|} = \frac{1}{|S\bar{V}_j|} + \frac{1}{|S\bar{V}_{j+1}|} - \frac{1}{|S\bar{U}_j|} - \frac{1}{|S\bar{U}_{j+1}|}\,, \tag{11}$$

or, equivalently,

$$\frac{\sin(\alpha+\beta)}{|SV_j|} - \frac{\sin(\alpha+\beta)}{|SU_j|} = \frac{\sin(\alpha+\beta)}{|S\bar{V}_j|} + \frac{\sin(\alpha+\beta)}{|S\bar{V}_{j+1}|} - \frac{\sin(\alpha+\beta)}{|S\bar{U}_j|} - \frac{\sin(\alpha+\beta)}{|S\bar{U}_{j+1}|}\,. \tag{12}$$

By similar derivations to that of (3) for the triangle SA_1C_1, we can obtain the following relations, using triangles SD_jB_1, SA_jC_1, SD_jB_{j+1}, $SD_{j+1}B_1$, SA_jC_{j+1}, $SA_{j+1}C_1$:

$$\frac{\sin(\alpha+\beta)}{|SV_j|} = \frac{\sin\alpha}{|SD_j|} + \frac{\sin\beta}{|SB_1|}\,,$$
$$\frac{\sin(\alpha+\beta)}{|SU_j|} = \frac{\sin\alpha}{|SC_1|} + \frac{\sin\beta}{|SA_j|}\,,$$
$$\frac{\sin(\alpha+\beta)}{|S\bar{V}_j|} = \frac{\sin\alpha}{|SD_j|} + \frac{\sin\beta}{|SB_{j+1}|}\,,$$
$$\frac{\sin(\alpha+\beta)}{|S\bar{V}_{j+1}|} = \frac{\sin\alpha}{|SD_{j+1}|} + \frac{\sin\beta}{|SB_1|}\,,$$
$$\frac{\sin(\alpha+\beta)}{|S\bar{U}_j|} = \frac{\sin\alpha}{|SC_{j+1}|} + \frac{\sin\beta}{|SA_j|}\,,$$
$$\frac{\sin(\alpha+\beta)}{|S\bar{U}_{j+1}|} = \frac{\sin\alpha}{|SC_1|} + \frac{\sin\beta}{|SA_{j+1}|}\,.$$

Hence, we can rewrite (12) as

$$\frac{\sin\alpha}{|SD_j|}+\frac{\sin\beta}{|SB_1|}-\frac{\sin\alpha}{|SC_1|}-\frac{\sin\beta}{|SA_j|}$$
$$=\frac{\sin\alpha}{|SD_j|}+\frac{\sin\beta}{|SB_{j+1}|}+\frac{\sin\alpha}{|SD_{j+1}|}+\frac{\sin\beta}{|SB_1|}-\frac{\sin\alpha}{|SC_{j+1}|}-\frac{\sin\beta}{|SA_j|}-\frac{\sin\alpha}{|SC_1|}-\frac{\sin\beta}{|SA_{j+1}|}\,,$$

from which we have the identity

$$\left(\frac{1}{|SA_{j+1}|}-\frac{1}{|SB_{j+1}|}\right)\sin\beta=\left(\frac{1}{|SD_{j+1}|}-\frac{1}{|SC_{j+1}|}\right)\sin\alpha\,,$$

or, equivalently,

$$\frac{|SB_{j+1}|-|SA_{j+1}|}{|SA_{j+1}|\,|SB_{j+1}|}\sin\beta=\frac{|SC_{j+1}|-|SD_{j+1}|}{|SD_{j+1}|\,|SC_{j+1}|}\sin\alpha\,. \tag{13}$$

Using the *Intersecting Chords Theorem*, we have $|SA_{j+1}|\,|SB_{j+1}| = |SC_{j+1}|\,|SD_{j+1}|$. And because

$$|SB_{j+1}|-|SA_{j+1}|=2\,|SO'|\;=2\,|SO|\sin\alpha\,,$$
$$|SC_{j+1}|-|SD_{j+1}|=2\,|SO''|=2\,|SO|\sin\beta\,,$$

we obtain, by (13),

$$\frac{2\,|SO|\sin\alpha}{|SC_{j+1}|\,|SD_{j+1}|}\sin\beta=\frac{2\,|SO|\sin\beta}{|SC_{j+1}|\,|SD_{j+1}|}\sin\alpha\,.$$

Clearly, this last equality is true. Hence, the equality (11) is true.

The induction hypothesis for j circles thus implies the validity of the *Multi-Butterfly Theorem* for $j+1$ circles. This completes the proof for all natural numbers n. □

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