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The Kelly Model for gambling and investing

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1 Introduction

In his book *A Mathematician Plays the Stock Market* [2], John Allen Paulos describes a scenario that occurred during the wild times when dotcom companies were going public on a daily basis. A certain investor is offered the following opportunity: Every Monday for a period of 52 weeks, the investor may invest funds in the stock of one dotcom company. On the ensuing Friday, the investor sells. The following Monday, he purchases new stock in another dotcom company. Each week, the value of the stock purchased has a probability of $\frac{1}{2}$ of increasing by 80%, and a probability of $\frac{1}{2}$ of decreasing by 60%, independently of what happened in previous weeks. This means that on average, the increase in value of the purchased stock is equal to

$$0.8 \times \frac{1}{2} - 0.6 \times \frac{1}{2} = 0.1\,,$$

giving an average return of 10% per week. The investor, who has a starting bankroll of ten thousand dollars to invest over a period of the coming 52 weeks, doesn't hesitate for a moment; he decides to invest the full amount, every week, in the stock of a dotcom company. After 52 weeks, it appears that our investor only has 2 dollars left of his initial ten-thousand-dollar bankroll. He is, quite literally, at a loss to figure it all out. But in fact, this investment result is not very surprising when you consider how dangerous it is to rely on averages in situations involving uncertainty. A person can drown, after all, in a lake that has an average depth of 25 cm. For situations involving uncertainty factors, you should never work with averages, but rather with probabilities! It is easily explained that the probability of nearly depleting the bankroll is large if the investor invests his whole bankroll in each transaction. The most likely path to develop over the course of 52 weeks is one in which the stock increases in value 50% of the time, and decreases in value 50% of the time. This path results in a bankroll of $1.8^{26} \times 0.4^{26} \times 10\,000 = 1.95$ dollars after 52 weeks. Running one hundred thousand simulations of the investments over 52 weeks renders a probability of about 50% that the investor's final bankroll will not exceed 1.95 dollars, and a small probability of 5.8% that the investor's final bankroll will be greater than his starting bankroll of ten thousand dollars.

Misled by seemingly favorable averages, our foolhardy investor stakes the full amount of his bankroll every week. Apparently, he is unacquainted with the *Kelly*

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strategy. According to this strategy, rather than investing the full amount of his current bankroll for every transaction, he would do better to invest a same fixed fraction of his current bankroll each time and to keep in reserve a fixed fraction of his current capital. The Kelly model will be discussed in the next sections. First, the case of a single betting object is analyzed and then that of multiple betting objects.

The Kelly model is named after the physicist John Kelly Jr. Working at Bell Labs, he published in 1956 a paper titled *A New Interpretation of Information Rate* [1] in the Bell System Technical Journal. Virtually no one took much note of the article when it first appeared. Nowadays it is widely used in gambling and investing. In the paper Kelly posited a scenario in which a horse-race better has an edge: a 'private wire' of somewhat reliable but not perfect tips from inside information. How should he bet? Wager too little, and the advantage is squandered. Too much, and ruin beckons. The Kelly bet size is found by maximizing the expected value of the logarithm of wealth, which is equivalent to maximizing the expected geometric growth rate.

2 The Kelly model with a single betting object

Consider the situation in which you can repeatedly make bets in a particular game with a single betting object. The game is assumed to be favorable to you, where favorable means that the expected value of the net payoff of the game is positive. For every dollar staked on a repetition of the game, you receive w_1 dollars back with probability p and w_2 dollars with probability $1-p$, where $0 < p < 1$, $w_1 > 1$ and $0 \leq w_2 < 1$. The outcomes of the successive games are assumed to be independent of each other. The key assumption for the Kelly betting model is as follows:

Assumption: *The parameters p, w_1 and w_2 satisfy*

$$pw_1 + (1-p)w_2 > 1 \qquad \text{and} \qquad pw_1 + (1-p)w_2 - 1 < (w_1 - 1)(1 - w_2)\,.$$

The first condition says that the game is favourable to you in terms of one-step expected value. It is noted that the first condition implies the second condition if $w_2 = 0$. You start with a certain bankroll, and it is assumed that you may stake any amount up to the size of your current bankroll each time. Then if you want to maximize the growth rate of your bankroll over the long run, the *Kelly formula* advises you to stake the following fixed fraction α of your current bankroll each time:

$$\alpha = \frac{pw_1 + (1-p)w_2 - 1}{(w_1 - 1)(1 - w_2)}\,. \qquad (1)$$

This formula will be derived in the next section. Note that, by the assumption made, $0 < \alpha < 1$. In the special case of $w_2 = 0$, the Kelly formula (1) reduces to

$$\alpha = \frac{pw_1 - 1}{w_1 - 1}\,, \qquad (2)$$

which can be interpreted as the ratio of the expected net gain per staked dollar and the payoff odds.

In the specific case of the investor with $p = 0.5$, $w_1 = 1.8$ and $w_2 = 0.4$, the Kelly strategy requires him to invest a fraction $\frac{5}{24}$ of his current bankroll for each transaction. In practical terms, this renders a practically zero probability of his ending with 1.95 dollars or less after 52 weeks. Simulation reveals that applying the Kelly strategy would give the investor about a 70% probability of ending with more than ten thousand dollars after 52 weeks, and about a 44% probability of ending with more than twenty thousand dollars.

The Kelly strategy was first used in casinos by mathematician Edward Thorp, in order to try out his winning blackjack system. Later, Thorp and a host of famous investors including Warren Buffett, successfully applied a form of the Kelly strategy to guide their stock market decisions.

3 Derivation of the Kelly formula

The strategy is to bet a fixed fraction α of your current bankroll each time, where $0 < \alpha < 1$. Here it is supposed that winnings are reinvested and that your bankroll is infinitely divisible. Letting V_0 be your starting bankroll, define the random variable V_m as

$$V_m = \text{the size of your bankroll after } m \text{ bets.}$$

For the mth bet, let the random variable W_m be equal to w_1 with probability p and be equal to w_2 with probability $1-p$. Noting that $V_m = (1-\alpha)V_{m-1} + \alpha V_{m-1}W_m$, it follows by induction that

$$V_m = \big(1 - \alpha + \alpha W_1\big) \times \cdots \times \big(1 - \alpha + \alpha W_m\big)V_0 \quad \text{for } m = 1, 2, \ldots.$$

In mathematics, a growth process is most often described with the help of an exponential function. This is the motivation to define the exponential growth factor G_m via the relationship

$$V_m = V_0 e^{mG_m},$$

where $e = 2.71828\ldots$ is the base of the natural logarithm. If you take the logarithm of both sides of this equation, then you see that the definition of G_m is equivalent to

$$G_m = \frac{1}{m} \ln\left(\frac{V_m}{V_0}\right).$$

Using the product formula for V_m and the fact that $\ln(ab) = \ln(a) + \ln(b)$, you find

$$G_m = \frac{1}{m}\Big(\ln\big(1 - \alpha + \alpha W_1\big) + \cdots + \ln\big(1 - \alpha + \alpha W_m\big)\Big).$$

Next, we apply the Law of Large Numbers, being one of the pillars of probability theory. Since the random variables $X_i = \ln(1 - \alpha + \alpha W_i)$ form a sequence of independent random variables having a common probability distribution, the Law of Large Numbers gives

$$\lim_{m\to\infty} G_m = \mathbb{E}\big[\ln(1 - \alpha + \alpha W)\big] \quad \text{with probability } 1,$$

where the random variable W is equal to w_1 with probability p and is equal to w_2 with probability $1-p$. Thus the long-run growth rate of your bankroll is equal to

$$\lim_{m\to\infty} G_m = p\ln(1-\alpha+\alpha w_1) + (1-p)\ln(1-\alpha+\alpha w_2) \quad \text{with probability } 1.$$

Putting the derivative of $g(\alpha) = p\ln(1-\alpha+\alpha w_1) + (1-p)\ln(1-\alpha+\alpha w_2)$ equal to 0, you get

$$\frac{p(w_1-1)}{1-\alpha+\alpha w_1} + \frac{(1-p)(w_2-1)}{1-\alpha+\alpha w_2} = 0.$$

This gives the formula (1) after a little algebra. Since the second derivative of $g(\alpha)$ is negative on $(0,1)$, the function $g(\alpha)$ is concave on $(0,1)$, and so $g(\alpha)$ attains its absolute maximum for the value of α in (1).

For the Kelly model with a single betting object, further results including central limit theorem type of results for V_m are discussed in Tijms [3].

4 Kelly betting with multiple betting objects

In investment situations and in sport events such as soccer matches and horse races, multiple investments or bets can be simultaneously made. Imagine that opportunities to bet or invest arise at successive times $t = 1, 2, \ldots$. There are n betting objects $j = 1, \ldots, n$, where $n \geq 2$. You can simultaneously bet on one or more of these objects.

Assumptions
(**a**) *At any betting opportunity, only one betting object can be successful (e.g., in a horse race only one horse can win), where object j will be successful with a given probability p_j and non-successful with probability $1-p_j$, independently of what happened at earlier betting opportunities. Hereby,*

$$0 < p_j < 1 \quad \textit{for all} j \qquad \textit{and} \qquad \sum_{j=1}^{n} p_j = 1\,.$$

(**b**) *At any betting opportunity, a stake on each non-successful object j is lost, while $f_j > 0$ dollars are added to your bankroll for every dollar staked on a successful object j. The payoffs f_j are such that $p_j f_j > 1$ for at least one object j and $\sum_{j=1}^{n} 1/f_j \geq 1$.*

The probabilities p_j are typically subjective probabilities being different for each person. For example, in horse racing you can imagine that your personal estimates of the win probability of the horses are different from the bookmaker's estimates. In the Assumptions, the requirement $\sum_{j=1}^{n} p_j = 1$ can be relaxed to $\sum_{j=1}^{n} p_j \leq 1$: to do that, introduce an auxiliary investment object $n+1$ with f_{n+1} being very close to 0 and let $p_{n+1} = 1 - \sum_{i=1}^{n} p_i$.

You start with a certain bankroll V_0 and the question is how to maximize the long-run growth rate of your bankroll. The Kelly betting strategy is now characterized by parameters $\alpha_1, \ldots, \alpha_n$ such that $\alpha_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \alpha_i \leq 1$. Under this strategy you stake the same fraction α_i of your current bankroll in object i each time, while you keep in reserve a fraction

$$\beta = 1 - \sum_{i=1}^{n} \alpha_i$$

of your current bankroll. Denote by V_m the size of your bankroll after the mth betting opportunity and let $G_m = \frac{1}{m}\ln(V_m/V_0)$ be the growth rate of your bankroll over the first m betting opportunities. Using again the law of large numbers, a generalization of the analysis in Section 3 leads to

$$\lim_{m\to\infty} G_m = \mathbb{E}\Big[\ln\big(\beta + \sum_{i=1}^{n} \alpha_i R_i\big)\Big] \quad \text{with probability } 1\,,$$

where the random vector $(R_1, \ldots, R_n)$ has the joint probability distribution

$$P(R_i = f_i,\, R_j = 0 \text{ for } j \neq i) = p_i \quad \text{for } i = 1, \ldots, n\,.$$

Thus, the long-run growth rate of your bankroll is equal to

$$\lim_{m\to\infty} G_m = \sum_{i=1}^{n} p_i\big(\ln(\beta + f_i\alpha_i)\big) \quad \text{with probability } 1\,. \tag{3}$$

The goal is to find the values for the α_i's such that the long-run growth rate of your bankroll is maximal. Thus, you have to solve the following optimization problem:

$$\begin{aligned}
\text{Maximize} \quad & f(\beta, \alpha_1, \ldots, \alpha_n) && = \sum_{i=1}^{n} p_i \ln(\beta + f_i\alpha_i) \\
\text{subject to} \quad & \beta + \sum_{i=1}^{n} \alpha_i && = 1 \\
& \beta, \alpha_1, \ldots, \alpha_n && \geq 0\,.
\end{aligned}$$

The objective function $f(\beta, \alpha_1, \ldots, \alpha_n)$ is concave on the convex set of feasible solutions of the optimization problem, as can be shown by analyzing the second-order partial derivatives of the function $\ln(x + y)$ in the two variables $x, y > 0$. An algorithm for the optimal values of β and the α_i's can be derived from the specific Kuhn-Tucker optimality conditions for a nonlinear optimization problem with linear constraints:

$$\begin{aligned}
\text{Maximize} \quad & f(x_1, \ldots, x_n) && && \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j && = b_i && \text{for } i = 1, 2 \ldots, m\,, \\
& x_j && \geq 0 && \text{for } j = 1, \ldots, n\,.
\end{aligned}$$

If the objective function $f(x_1, \ldots, x_n)$ is differentiable and concave on the convex set of feasible solutions of the optimization problem, then a basic result in the theory of nonlinear optimization says that a global maximum is attained for the feasible solution $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ if Lagrangian multipliers $\lambda_1^*, \ldots, \lambda_m^*$ exist satisfying the Kuhn-Tucker conditions

$$\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* a_{ij} \leq 0 \qquad \text{for } j = 1, \ldots, n\,,$$

$$x_j^* \Big(\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* a_{ij} \Big) = 0 \qquad \text{for } j = 1, \ldots, n\,,$$

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \qquad \text{for } i = 1, \ldots, m\,,$$

$$x_j^* \geq 0 \qquad \text{for } j = 1, \ldots, n\,.$$

By these Kuhn-Tucker conditions, non-negative values $\beta, \alpha_1, \ldots, \alpha_n$ satisfying

$$\beta + \sum_{i=1}^{n} \alpha_i = 1$$

provide an optimal solution for the Kelly optimization problem if, for some real number λ,

$$\frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda \ \ \leq 0 \ \text{ for } i = 1, \ldots, n\,, \qquad \sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} - \lambda \ \ \leq 0\,,$$

$$\alpha_i \Big(\frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda \Big) = 0 \ \text{ for } i = 1, \ldots, n\,, \qquad \beta \Big(\sum_{i=1}^{n} \frac{p_i}{\beta + f_i \alpha_i} - \lambda \Big) = 0\,.$$

The first and the third condition give

$$\frac{p_i f_i}{\beta} - \lambda \leq 0 \ \text{ if } \alpha_i = 0 \quad \text{ and } \quad \frac{p_i f_i}{\beta + f_i \alpha_i} - \lambda = 0 \ \text{ if } \alpha_i > 0\,.$$

The key step is to assume that $\beta > 0$ in the optimal solution; that is, a positive proportion of the bankroll is always kept in reserve. This premise is reasonable in view of $p_j < 1$ for all j. Taking $\beta > 0$, the fourth condition becomes $\sum_{i=1}^{n} p_i/(\beta + f_i \alpha_i) = \lambda$, implying the second condition. Also, using the condition $\sum_{i=1}^{n} \alpha_i = 1 - \beta$, you easily find that

$$\beta = \frac{1 - \sum_{i \in V} p_i/\lambda}{1 - \sum_{i \in V} 1/f_i} \quad \text{ with } V = \{i \mid \alpha_i > 0\}\,.$$

Next is matter of some manipulations to get $\lambda = 1$ and to arrive at the algorithm for the optimal values of the α_i's. We omit the technical details and merely give the final algorithm.

Algorithm

Step 0. Renumber the indexes such that $p_1 f_1 \geq p_2 f_2 \geq \cdots \geq p_n f_n$.

Step 1. Let r be the largest integer k for which $\sum_{j=1}^{k} 1/f_j < 1$.

Step 2. Calculate for each index $k = 1, \ldots, r$ the number

$$B(k) = \frac{1 - \sum_{j=1}^{k} p_j}{1 - \sum_{j=1}^{k} 1/f_j}$$

Let s be the first index k for which $p_{k+1} f_{k+1} < B(k)$, and let $\beta = B(s)$.

Step 3. Set $\alpha_i = p_i - \beta/f_i$ for $i = 1, \ldots, s$ and $\alpha_i = 0$ for $i > s$.

Index r satisfies $r < n$ by part (**b**) of the Assumptions. This implies that $B(s) > 0$. Therefore, $\alpha_i < p_i$ for all i and so $\sum_{i=1}^{n} \alpha_i < 1$. This verifies that $\beta = 1 - \sum_{i=1}^{n} \alpha_i > 0$; that is, a positive fraction of your bankroll is kept in reserve each time. It is also noted that the algorithm with $n = 1$ results in the optimal betting fraction $(p_1 f_1 - 1)/(f_1 - 1)$, in agreement with the Kelly formula (2) for the case of a single betting object.

Next we give two numerical examples to illustrate the algorithm.

5 Numerical examples

The Kelly strategy has been developed for situations in which many betting opportunities repeat themselves under identical conditions. However, the Kelly strategy provides also a useful heuristic guideline for situations with only one betting opportunity.

Example 1 (*soccer*)**.** Suppose that the soccer club Manchester United is hosting a match against Liverpool, and that a bookmaker is paying out 4.5 times the stake if Liverpool wins, 4.5 times the stake if the match ends in a draw, and 1.75 times the stake if Manchester United wins. You estimate Liverpool's chance of winning at 25%, the chance of the game ending in a draw at 25%, and the chance of Manchester winning at 50%. If you are prepared to bet 100 dollars, then how should you bet on this match? The Kelly betting model with $n = 3$ betting objects applies, where

$$\begin{aligned} &p_1 = 0.25 \text{ (win Liverpool)}, \\ &p_2 = 0.25 \text{ (draw)}, \\ &p_3 = 0.50 \text{ (win United)}, \\ &f_1 = f_2 = 4.5 \text{ and } f_3 = 1.75\,. \end{aligned}$$

Since $p_1 f_1 = p_2 f_2 = 1.125$ and $p_3 f_3 = 0.875$, the condition $p_1 f_1 \geq p_2 f_2 \geq p_3 f_3$ is satisfied. The algorithm goes as follows:

Step 1. $r = 2$ since $1/f_1 = \frac{10}{45}$, $1/f_1 + 1/f_2 = \frac{20}{45}$ and $1/f_1 + 1/f_2 + 1/f_3 > 1$.
Step 2. $B(1) = \frac{27}{28}$, $B(2) = \frac{9}{10}$ and $p_2 f_2 = 1.125 > B(1)$.
This gives $s = 2$ with $\beta = B(s) = 0.9$.
Step 3. $\alpha_1 = \alpha_2 = 0.25 - \frac{0.9}{4.5} = 0.05$ and $\alpha_3 = 0$.

Thus, the Kelly strategy proposes that you stake 5% of your bankroll of 100 dollars on a win for Liverpool, 5% on a draw, and 0% on a win for Manchester United. For this strategy, the subjective expected value of your bankroll after the match is equal to $100 - 10 + 0.25 \times 4.5 \times 5 + 0.25 \times 4.5 \times 5 = 101.25$ dollars. The expected value of the percentage increase of your bankroll is 1.25%. It is interesting to note that the two concurrent bets on the soccer match act as a partial hedge for each other, reducing the overall level of risk.

Example 2 (*horse race*)**.** In a horse race, there are seven horses A, B, C , D , E, F and G with respective win probabilities 40%, 25%, 20%, 7%, 4%, 3% and 1% and payoff odds 1.625:1, 2.9:1, 4.5:1, 9:1, 14:1, 17:1 and 49:1. Payoff odds a:1 means that in case of a win you will receive your stake plus a dollars for each dollar staked. Numbering the horses as 1 ($= C$), 2 ($= A$), 3 ($= B$), 4 ($= D$), 5 ($= E$), 6 ($= F$), and 7 ($= G$), the Kelly model applies with

$$\begin{array}{lllllll} p_1 = 0.2 & p_2 = 0.4 & p_3 = 0.25 & p_4 = 0.07 & p_5 = 0.04 & p_6 = 0.03 & p_7 = 0.01 \\ f_1 = 5.5 & f_2 = 2.625 & f_3 = 3.9 & f_4 = 10 & f_5 = 15 & f_6 = 18 & f_7 = 50 \end{array}$$

satisfying the condition of decreasing $p_i f_i$ values:

$$\begin{aligned} p_1 f_1 &= 1.1 \\ p_2 f_2 &= 1.05 \\ p_3 f_3 &= 0.975 \\ p_4 f_4 &= 0.7 \\ p_5 f_5 &= 0.6 \\ p_6 f_6 &= 0.54 \\ p_7 f_7 &= 0.50\,. \end{aligned}$$

The algorithm goes as follows:

Step 1. The index $r = 5$ is the largest value of k for which $\sum_{j=1}^{k} 1/f_j < 1$.
Step 2. $B(1) = 0.9778$, $B(2) = 0.9149$, $B(3) = 0.8296$, $B(4) = 0.9899$ and $B(5) = 2.8264$.
Also, $p_2 f_2 > B(1)$ and $p_3 f_3 > B(2)$ but $p_4 f_4 \leq B(3)$.
This gives $s = 3$ with $\beta = B(s) = 0.8296$.
Step 3. $\alpha_1 = 0.0492$, $\alpha_2 = 0.0840$, $\alpha_3 = 0.0373$ and $\alpha_j = 0$ for $j > 3$.

Thus you bet 8.4% of your bankroll on horse A, 3.7% on horse B, 4.9% on horse C and nothing on the other horses. It is noteworthy that horse B is included in your bet, even though a bet on horse B alone is not favorable ($p_3 f_3 < 1$). The expected value of the percentage increase of your bankroll is $100 \times \sum_{j=1}^{3} (p_i f_i \alpha_i - \alpha_i) = 4.6\%$.

References

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[2] J.A. Paulos, *A Mathematician Plays The Stock Market*, Basic Books, 2004.

[3] H.C. Tijms, *Understanding Probability*, third edition, Cambridge University Press, 2012.