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Squaring the circle like a medieval master mason

Frédéric Beatrix[1]

1 Context

"Squaring the circle" is a classic math problem in geometry. It is one of the three great problems of antiquity, the other two being the trisection of the angle and the duplication of the cube. It is the challenge of constructing, *using solely a compass and straightedge*, a square with the same area as a given circle. This essentially comes down to constructing a square with side lengths $\sqrt{\pi}$. The rules of construction are as follows:

- R1 We are initially given just two points.
  The distance between these points defines 1 unit.
- R2 We are allowed to draw a straight line through any two given points.
  This counts as one step.
- R3 We are allowed to draw a circle with centre in a given point that intersects another given point. *This counts as one step.*
- R4 Each intersection of a drawn line or circle with another line or circle defines a new given point. *This is not counted as a step.*
- R5 We are allowed only finitely many steps.

In 1837, Pierre Wantzel [1] proved that only lengths which are algebraic numbers can be constructed with compass and straightedge. More precisely, the constructed numbers turn out to be all the numbers that you can calculate in a finite number of steps using the four arithmetic operations $+$, $-$, $*$, $/$ as well as taking square roots $\sqrt{\ }$. This is the reason why we can construct for example $\sqrt{2}$ which is the diagonal of a 1×1 square □, or $\sqrt{5}$ which is the diagonal of a *"Quadratum Lungum"*, a 1×2 rectangle ▭.

In 1882, Ferdinand von Lindemann [2] proved that π is not an algebraic number since it is not the root of a non-zero polynomial of finite degree with rational coefficients; we call such numbers *transcendental*. Therefore, squaring the circle is impossible: we cannot construct $\sqrt{\pi}$ using solely a compass and straightedge.

Since then, mathematicians have endeavoured to provide geometric approximations of $\sqrt{\pi}$, though all approximate constructions to date are borne from complex figures requiring multiple construction steps. Theoretical accuracy is achieved through complexity. In the context of a drawing by hand *using solely a compass and straightedge*, physical accuracy rapidly dissolves when performing an increasing number of construction steps due to the embedded error of drawing tools and the draughtsperson. In order to propose an adequate answer to the original problem, one has to consider that the greater level of accuracy is actually obtained with a greater level of simplicity.

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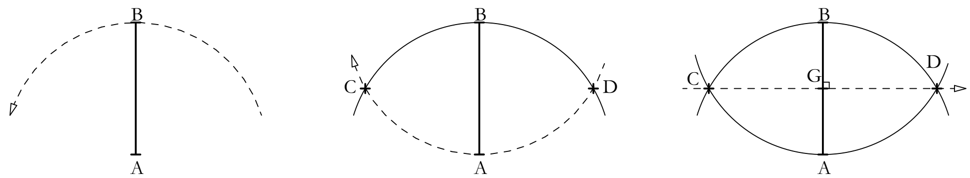
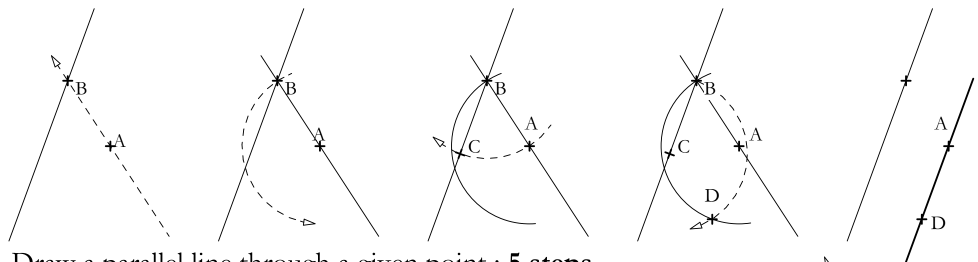
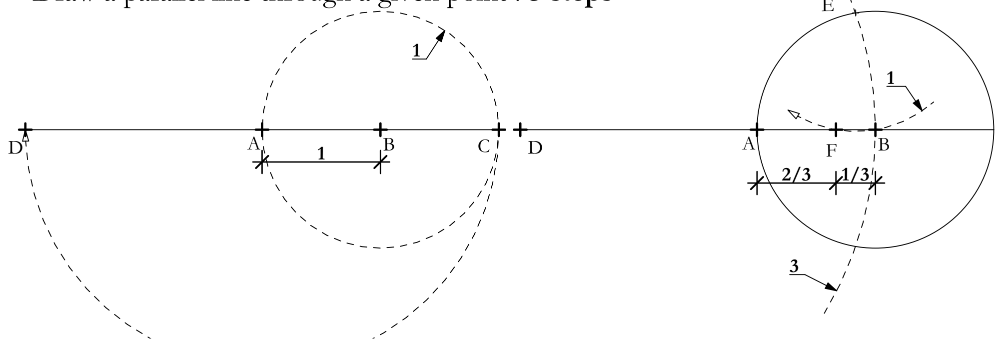


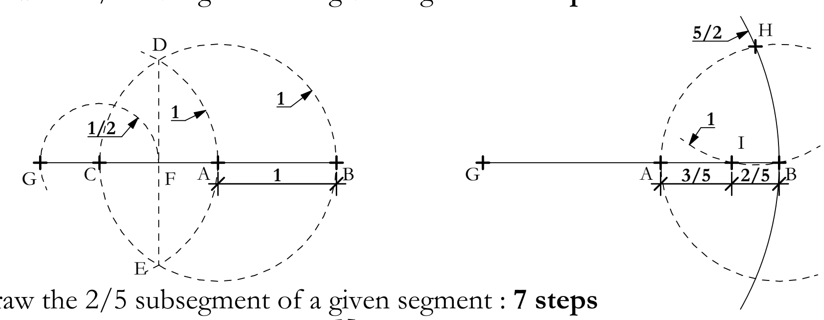
Draw a perpendicular line or find the middle of a given segment : **3 steps**



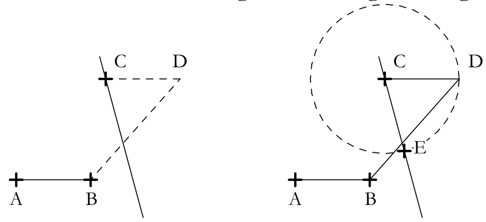
Draw a parallel line through a given point : **5 steps**



Draw the 1/3 subsegment of a given segment : **4 steps**



Draw the 2/5 subsegment of a given segment : **7 steps**



Find a point E on the line through C such that |CE|=|AB| : **11 steps**

Figure 1: Steps for some basic constructions

2 Definitions

ACCURACY increases as the relative error of the constructed length to $\sqrt{\pi}$ decreases.
ELEGANCE reflects the graphic beauty of the constructed length approximating $\sqrt{\pi}$.
SIMPLICITY is measured by number of steps leading to the length approximating $\sqrt{\pi}$. It corresponds to the "accuracy factor" as defined by Emile Lemoine [3]. Should we consider an incremental error of one millimetre per ten metres (0.01%) for each hand-drawn *STEP*, then the cumulative error after ten steps is about 0.1%.
STEPS are defined as per the following examples, see *Figure 1*:

0 step: A given figure
1 step: Draw a line through two given points (R2).
1 step: Draw a circle with centred in a given point and through a given point (R3).
3 steps: Draw a perpendicular line to, or find the middle of, a given line segment.
5 steps: Draw a line parallel to a given line through a given point.

4 steps: Trisect a given segment; that is, draw a $\frac{1}{3}$rd subsegment of the segment AB.
1: Draw the circle with centre B and radius |AB| to get C.
2: Draw the circle with centre A and radius |AC| to get D.
3: Draw the circle with centre D and radius |DB| to get E.
4: Draw the circle with centre E and radius |EB| to get F.
Then $|BF| = \frac{1}{3}|BA|$.

7 steps: Draw a $\frac{2}{5}$th subsegment of a given segment AB.
1: Draw the circle with centre A with radius |AB| to get C.
2: Draw the circle with centre C with radius |CA| to get D and E.
3: Draw the line DE which intersects AB at F.
4: Draw the circle with centre C with radius |CF| to get G.
5: Draw the circle with centre B with radius |BA|.
6: Draw the circle with centre G with radius |GB| to get H.
7: Draw the circle with centre H with radius |HB| to get I.
Then $|BI| = \frac{2}{5}|BA|$.

11 steps: Find a point E on the line through C such that $|CE| = |AB|$.
We want the point E on the given line through C that satisfies $|CE| = |AB|$.
1: Draw the parallel line to AB through C. (4 steps)
2: Draw the parallel line to AC through B. The intersection is D. (4 steps)
3: Draw the circle with centre C and radius |CD| to get E.
Then $|CE| = |CD| = |AB|$.

3 Accuracy vs. Elegance and Simplicity

In 1913, Srinivasan Ramanujan [5] proposed a complex figure using $\frac{355}{113}$ to approximate π correctly to 6 decimal places. Then, in 1914, he proposed another complicated construction, giving an approximation of π with

$$\sqrt[4]{9^2 + \frac{19^2}{22}},$$

correct to 8 decimal places in theory; see Figure 2. Though operatively, with a figure requiring 58 construction steps, the error on the hand-drawn segment is about 0.6% which annihilates the level of theoretical accuracy.

1: We have given the points A and O which define the unit length $|OA| = 1$.
2: Draw a line through A and O. (1 step)
3: Draw the circle centred in O with radius $|OA| = 1$, giving point B. (1 step)
4: Bisect AB with a line, giving intersection point C. (3 steps)
5: Trisect AO at T. (4 steps)
6: Draw the line through B and C. (1 step)
7: Find point M on the line through B and C with $|CM| = |AT|$. (11 steps)
8: Draw the circle with centre M and radius |MC| to get N. (1 step)
9: Draw a line through A and M and a line through A and N. (2 steps)
10: Draw a circle with centre A and radius |AM| to get P. (1 step)
11: Draw a line through P parallel to MN to get Q. (5 steps)
12: Draw a line through O and Q. (1 step)
13: Draw a line through T parallel to OQ, to get R. (6 steps)
14: Draw a line perpendicular to AO through A. (3 steps)
15: Draw a circle with centre A and radius |AR| to find the point S. (1 step)
16: Draw a line through O and S. (5 steps)
We have so far constructed $|OS| \approx \frac{\pi^2}{9}$.
17: Draw a circle with centre O and radius |OB| to get V. (1 step)
18: Bisect SV to get O′, and draw the circle with centre O′ and radius |O′S|. (4 steps)
19: Draw the line perpendicular to SV through O, to get W. (3 steps)
20: $|OW| \approx \frac{\pi}{3}$ so draw circles to find X with $|OX| = 3|OW| \approx \pi$. (2 steps)
21: Create Y so that $|XY| = |OX| + 1$. (1 step)
22: Bisect XY to find O″. (3 steps)
23: Draw the circle with centre O″ and radius |OX| to get Z. (1 step)
Then $|OZ| \approx \sqrt{\pi}$.

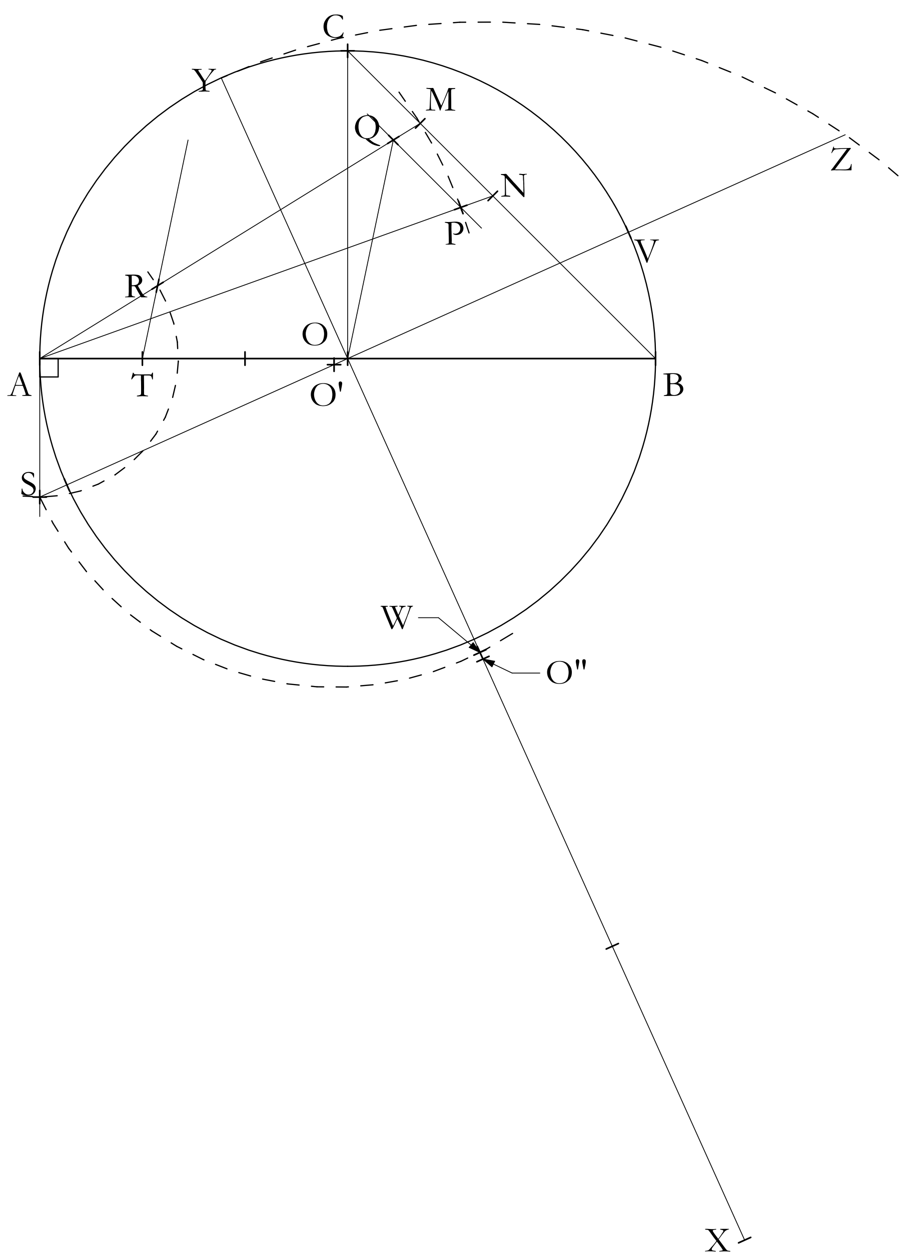


Figure 2: Construction by Ramanujan (1914), replicated from [4].
$ACCURACY$: ■■■■■ $ELEGANCE$: □□□□□ $SIMPLICITY$: ■□□□□

The latest geometric construction to approximate π dates from 3rd August 2019: Hùng Việt Chu [6] proposed a construction providing an approximation of π by

$$\sqrt{\frac{63}{25}\left(1+\frac{5}{2}\frac{15\sqrt{5}-7}{269}\right)}$$

which is, in theory, correct to 9 decimal places but requires at least 68 steps to draw by hand. As a result, the end segment, drawn with a straightedge and compass, has an error of about 0.7% which annihilates the expected level of accuracy; see Figure 3.

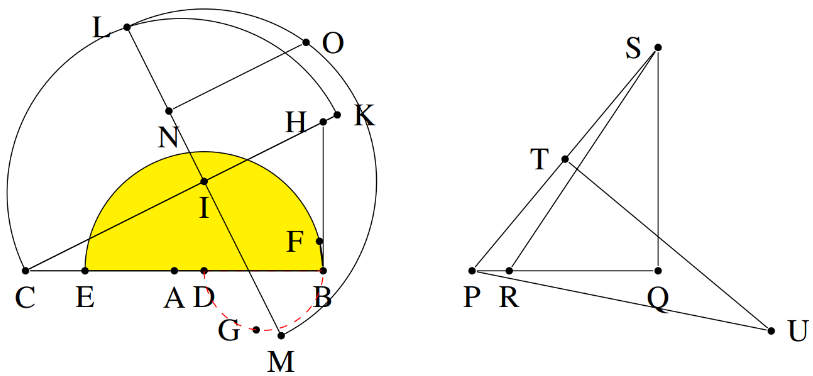


Figure 3: Construction by Hùng Việt Chu (2019), from [6].

$ACCURACY$: ■■■■■ $ELEGANCE$: □□□□□ $SIMPLICITY$: □□□□□

In the same publication, Hùng Việt Chu also proposed a much simpler construction based on the approximation that he attributes to Robert Dixon [7]:

$$\frac{6\varphi^2}{5} \approx \pi \qquad \text{where} \qquad \varphi = \frac{\sqrt{5}+1}{2}\,.$$

The beauty of this approximation is to connect π with the "Golden Ratio" φ.

The Golden Ratio φ (phi) is occasionally found in nature, for example in the spirals on the pineapple, the artichoke or the pine cone (13 spirals in one direction and 8 in the other), or the spiral of seeds of the sunflower (13 in one direction and 21 in the other direction). These numbers are part of the famous Fibonacci sequence where each new term is the sum of the previous two:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\,.$$

And indeed, the ratio of successive Fibonacci terms converges to the Golden Ratio φ.

Here are the proposed 25 steps:

1: We have given the points A and B which define the unit length $|AB| = 1$.
2: Draw the line perpendicular to AB through B. (4 steps)
3: By drawing circles, find the point C on that line satisfying $|BC| = 2$. (2 steps)
4: Draw the circle centered at A with radius $|AC|$, to get D. (1 step)
5: Find M on line through A and D such that $|DM| = \frac{2}{5}|AD|$. (7 steps)
6: Let N be the far point on this line for which $|ND| = \frac{1}{2}|DM|$. (4 steps)
7: Bisect NB to a point to use as centre for a circle intersecting N and B. (4 steps)
8: Find the line perpendicular to NB through M, to get H. (3 steps)
Then $|MH| = \sqrt{\frac{6\varphi^2}{5}}$.

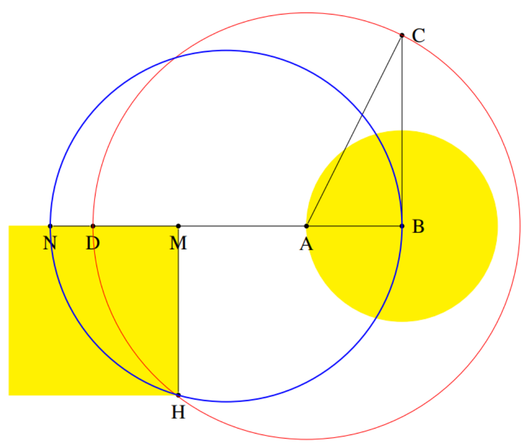


Figure 4: Construction by Hùng Việt Chu with Dixon approximation (2019) from [6]
$ACCURACY$: ■■■■□ $ELEGANCE$: ■■■□□ $SIMPLICITY$: ■■■□□

4 Golden Ratio and architecture

There have been many conjectures about the possible presence of φ in ancient Greek and Egyptian architecture. Some coincidences have lead to this popular belief. For example, the Great Pyramid at Giza has lengths with ratio roughly $\frac{14}{11}$ which is an approximation of $\sqrt{\varphi}$ inducing close approximations of φ in its proportions.

The Parthenon is also a common target, and the symbol φ, or "Phi", for the Golden Ratio was named after its sculptor and architect Phidias ($\varphi\epsilon\iota\delta\iota\alpha\zeta$). Many pseudo-archaeologists have tried their best to stick some golden rectangles on its elevation. They unfortunately miss in the process three essential facts. Firstly, Euclid defines the φ as a linear proportion – not a golden rectangle. Secondly, the arithmetic of φ requires the numbers 1, 2 and $\sqrt{5}$ which are naturally found in a *Quadratum Lungum* (double square) as shown in Figure 6. And finally, architects first design the floor plan, and the dimensions and proportions of the elevation are a consequence.

In a coming book, I will demonstrate the geometrical layout – with straight edge and compass – which defines the floor plan of the Parthenon. It does indeed require a

Quadratum Lungum, and, as a side-effect, it does arguably include φ as a linear proportion as shown in Figure 5 .

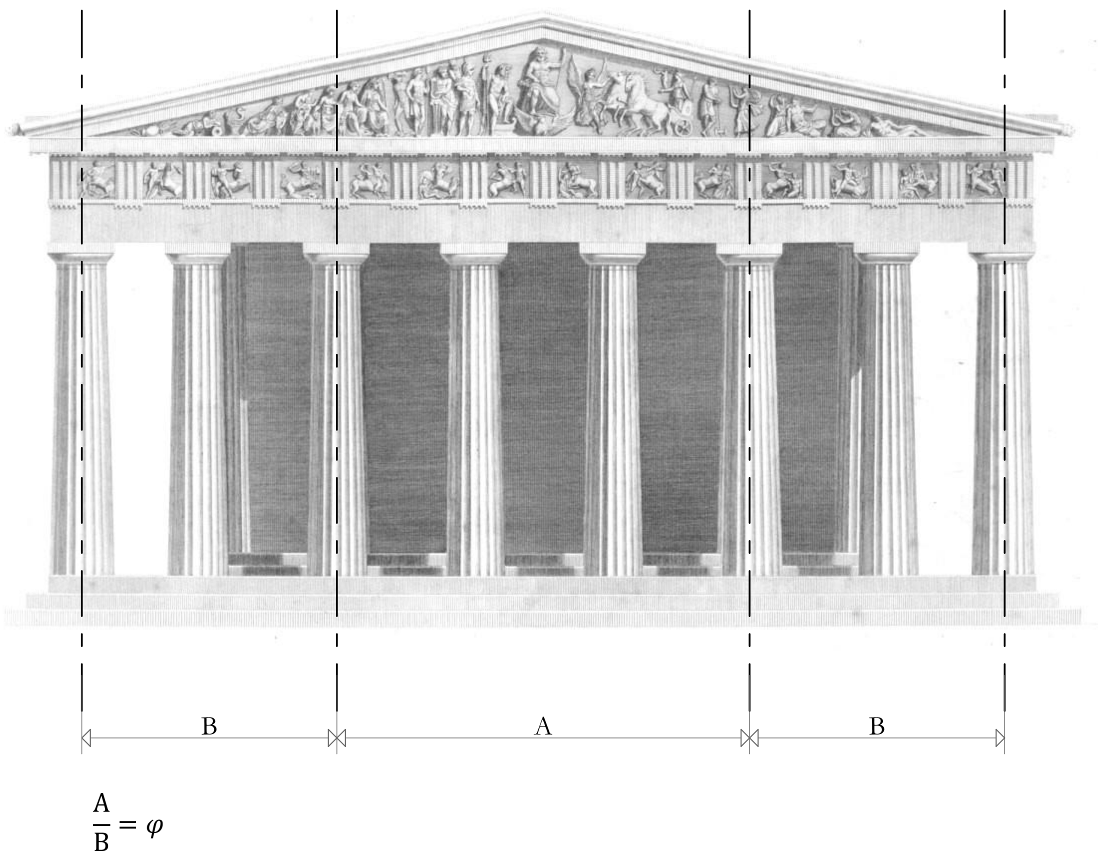


Figure 5: Parthenon oriental elevation $\frac{A+B}{A} = \frac{A}{B} = \varphi$ [8]

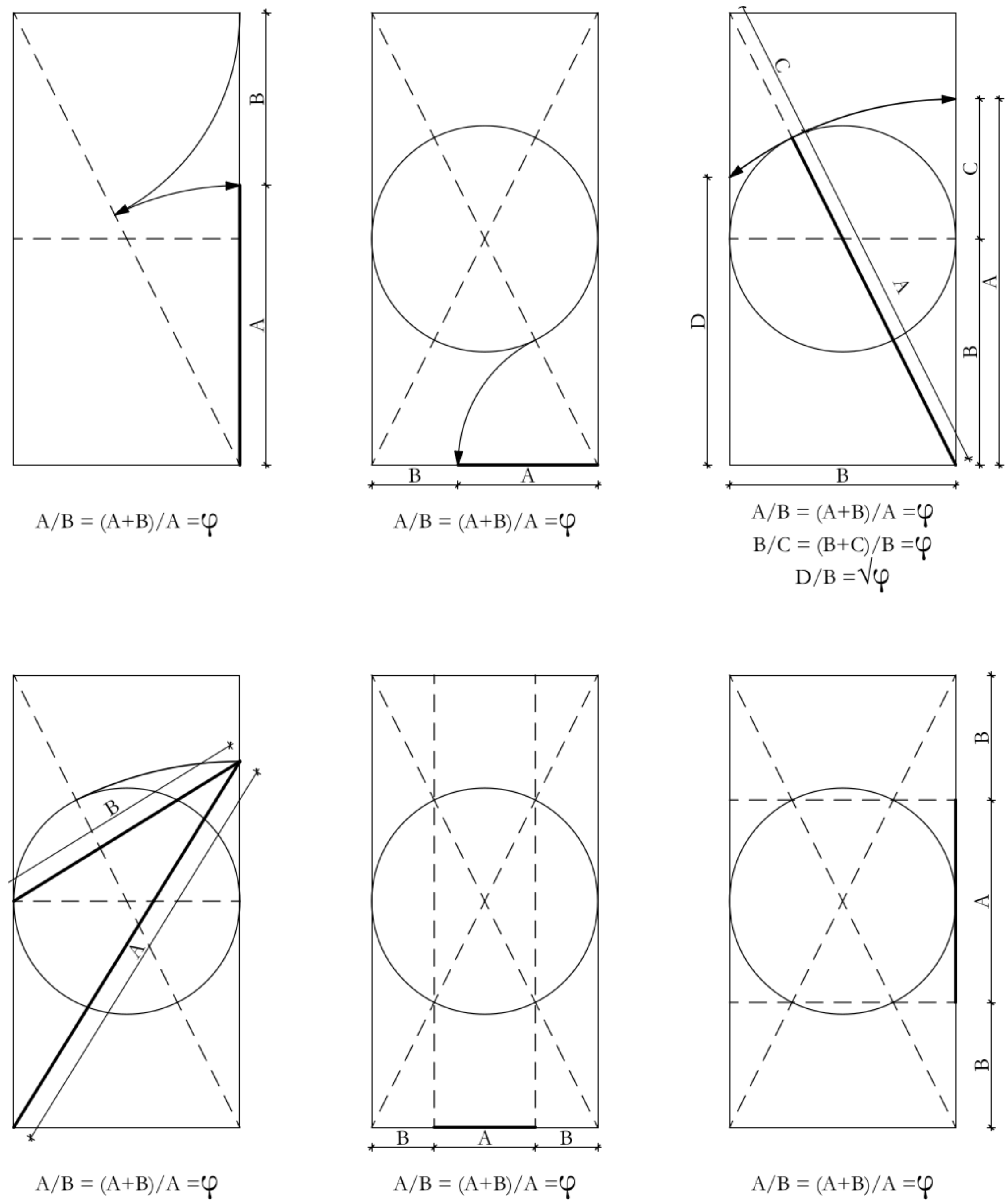


Figure 6: Multiple instances of the Golden Ratio inside the *quadratum lungum*

5 Proposed dating for the approximation $\frac{6\varphi^2}{5} \approx \pi$

The approximation $\frac{6\varphi^2}{5} \approx \pi$ may operatively (not arithmetically) date from early 12th century since it is a geometric quality of the building units used by the French builders of gothic cathedrals [2]. At the time, the unit is the *Ligne du Roi de France* and the standard set of French medieval units are the following:

34 *lignes* =	7.64cm	is one *Palmus minor*
55 *lignes* =	12.36cm	is one *Palmus major*
89 *lignes* =	20.00cm	is one *Empan* (span)
144 *lignes* =	32.36cm	is one *pied* (foot)
233 *lignes* =	52.36cm	is one French medieval *Coudée* (Cubit).

These are all numbers of the Fibonacci sequence. More specifically, if you consider the *Coudée* (233 *lignes*) C and the *Empan* (89 *lignes*) E, then we have $C \approx E\varphi^2$.

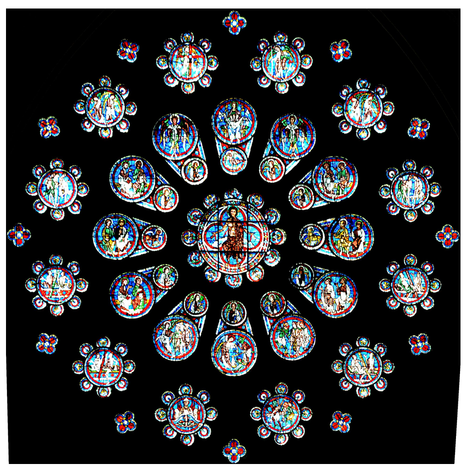


Figure 7: 12-petal Rose Window of the western façade of Chartres Cathedral [9]

Furthermore, a Master Mason at work will necessarily find out that a French Medieval *Coudée* measures 1/6 of the perimeter of a circle with diameter 5 *Empan*. Indeed, this property allows him to easily construct a large-scale 6-petal or 12-petal "rose window", as you commonly find in the axis of the west elevation of so many cathedrals, such as the Rose Window on the occidental façade of Chartres Cathedral, as shown in Figure 7. These operative qualities translate mathematically as $5E\pi \approx 6C$, or $6\varphi^2 \approx 5\pi$, at 0.0015% which is near-perfect for operative masonry[3].

[2] The French heritage authority *Centre des Monuments Nationaux* mentions the presence of these units at the *Abbaye du Thoronet* which dates from 1136 [10]

[3] In medieval times, "operative masonry" was roughly what we now call "architecture".

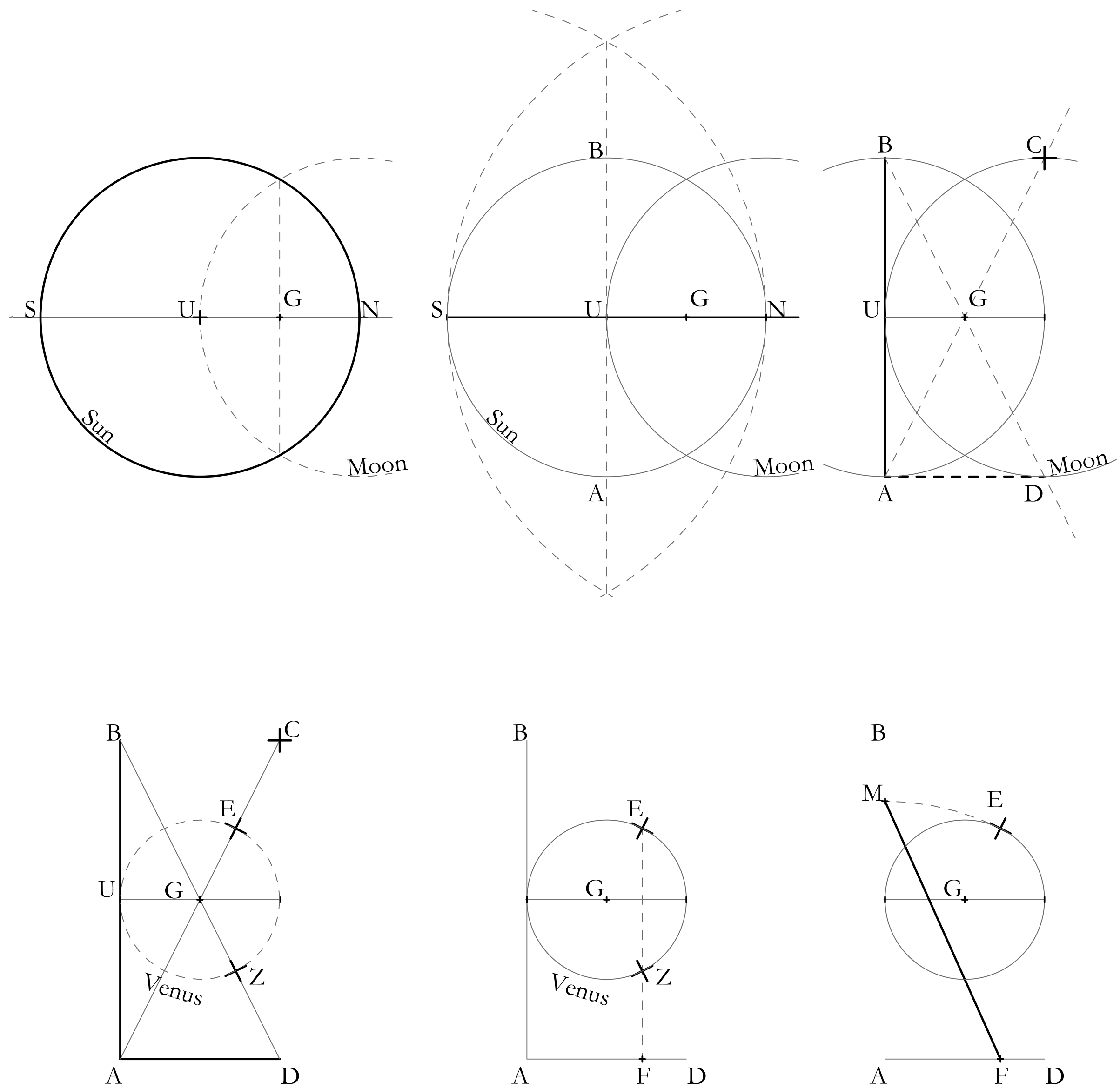


Figure 8: Squaring the circle in 13 steps: $|\mathrm{UN}| = |\mathrm{AD}| = 1$ and $|\mathrm{FM}| \approx \sqrt{\pi}$

$ACCURACY$: ■■■■□ $ELEGANCE$: ■■■■■ $SIMPLICITY$: ■■■■■

6 Squaring the circle like a medieval Master Mason

With the medieval approximation that we have presented, we propose a simple, elegant and accurate method for squaring the circle; see Figure 5.

0: We have given the points U and N which define the unit length $|\text{UN}| = 1$.
1: Draw the line through U and N. (1 step)
2: Draw the circle "Sun" with centre U and radius $|\text{UN}| = 1$. (1 step)
3: Draw the circle "Moon" with centre N and radius $|\text{UN}| = 1$. (1 step)
4: Draw the intersection line between "Sun" and "Moon" to get G. (1 step)
5: Bisect SUN to get A and B. (3 steps)
6: Draw the line through A and G which intersects "Moon" at C. (1 step)
7: Draw the line through B and G which intersects "Moon" at D. (1 step)
ABCD is then a *Quadratum Lungum*.
8: Draw the line through D and A. (1 step)
9: Draw the circle "Venus" with centre G and radius |GU| to get E and Z. (1 step)
10: Draw EZ which intersects AD at F (1 step)
11: Draw the arc centred at A, radius |AE|, which cuts AB at M (1 step).
Then $|\text{FM}| = \sqrt{\varphi^2 + (\frac{\varphi}{\sqrt{5}})^2} \approx \sqrt{\pi}$.

The accuracy of this approximation is 0.0007% - that is 7mm over one kilometre. The level of accuracy is the same as the approximation in Figure 4 by Hùng Việt Chu. However, our construction requires only 13 steps and produces a very neat and elegant figure. For hand-drawn figures, this is the best solution for approximately squaring the circle to date.

Proof. The length of the hypotenuse of ADC is $|\text{AC}| = \sqrt{AD^2 + DC^2} = \sqrt{5}$. Therefore, $|\text{AM}| = |\text{AE}| = |\text{AG}| + |\text{GE}| = \frac{\sqrt{5}}{2} + \frac{1}{2} = \varphi$. Since $\frac{AE}{AC} = \frac{AF}{AD}$, it follows that $|\text{AF}| = \frac{\varphi}{\sqrt{5}}$. The hypotenuse of FAM is $|\text{FM}| = \sqrt{\varphi^2 + (\frac{\varphi}{\sqrt{5}})^2} = \sqrt{\frac{6\varphi^2}{5}} \approx \sqrt{\pi}$, as desired. □

7 Bonus: The pentagram hidden in *Quadratum Lungum*

Also from Figure 4, we easily calculate that $|\text{MD}| = \sqrt{\varphi + 2}$ and $|\text{MO}| = \sqrt{3 - \varphi}$. Thus, $\frac{|\text{MD}|}{|\text{MO}|} = \varphi$; see Figure 9. These dimensions are respectively the isosceles side |MD| and the base |MO| of a "golden triangle" inscribed inside a circle with radius 1. Therefore, you can immediately build a regular pentagon and pentagram inside the "moon".

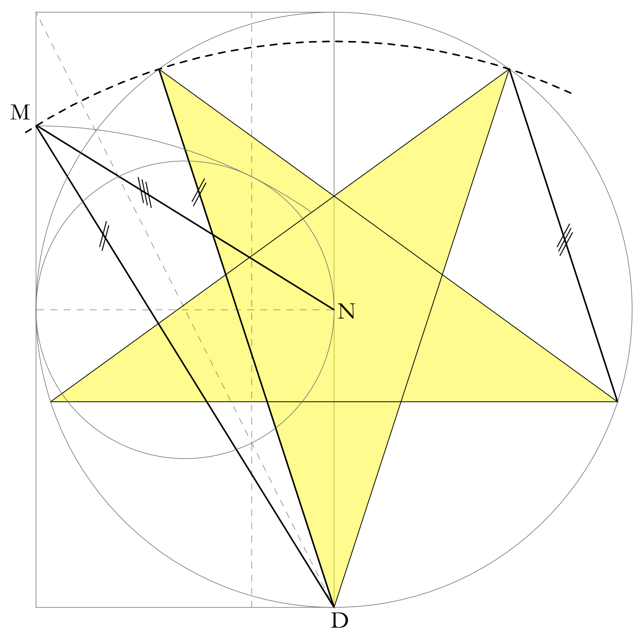


Figure 9: The pentagram hidden in *Quadratum Lungum*

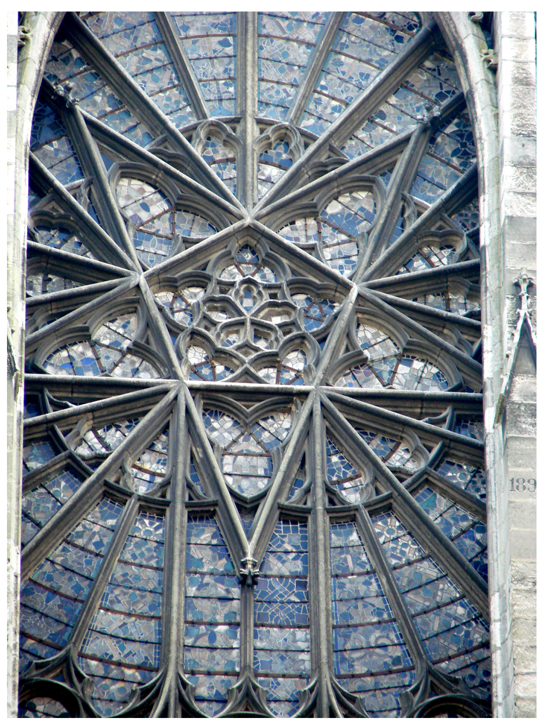


Figure 10: Example of a pentagram in a rose window, Amiens Cathedral [11]

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