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Infinite series for $\pi/3$ and other identities

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1 Introduction

Using methods from calculus, we combine classical identities for π, $\ln 2$, and harmonic numbers, to derive a nice infinite series formula for $\pi/3$ that does not appear to be well known. In addition, we give twenty-seven related identities involving π and other irrational numbers.

2 Main identity and proof

Recall the identity known in the literature as the Gregory–Leibniz Formula for π [1]:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots . \tag{1}$$

This identity is immediate from the Maclaurin series expansion of $\arctan x$ at $x = 1$. We prove another infinite series formula that can be used to compute the value of π.

Theorem 1. *We have the identity*

$$\frac{\pi}{3} \;=\; \sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-3)} .$$

Proof. We begin with the Maclaurin series for the natural logarithm of 2:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) = \ln 2 . \tag{2}$$

We will use the limit laws from calculus. Rewrite (2) as

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \ln 2 ;$$

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thus

$$\sum_{n=1}^{\infty} \frac{1}{4n(4n-2)} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \frac{1}{4} \ln 2 \,. \tag{3}$$

Euler [2] found the difference between the kth harmonic number and $\ln k$ approaches a constant $\gamma = 0.5772\ldots$ (the so-called *Euler-Mascheroni constant*), so we have

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - \ln(2n)\right) = \gamma. \tag{4}$$

By adding the limits from Equations (2) and (4), dividing by 2 and writing $\ln(2n) = \ln 2 + \ln n$, we find

$$\lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2}(\ln 2 + \ln n)\right) = \frac{1}{2}(\ln 2 + \gamma) \,.$$

Thus,

$$\lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2} \ln n\right) = \ln 2 + \frac{1}{2}\gamma \,. \tag{5}$$

By splitting the sum on the left in half and reorganising, Equation (5) may be rewritten

$$\begin{aligned} &\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{2n-1}\right) \qquad (6) \\ &= \ln 2 + \frac{1}{2}\gamma - \lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{2} \ln n\right) \,. \end{aligned}$$

Now, substitute $n/2$ for n in (5), then subtract $\frac{1}{2} \ln 2$ from both sides, to arrive at

$$\frac{1}{2} \ln 2 + \frac{1}{2}\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{2} \ln n\right) \,. \tag{7}$$

Adding the corresponding sides of (6) and (7), then subtracting $\frac{1}{2} \ln 2 + \gamma/2$ from the resulting equation, gives

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+3} + \cdots + \frac{1}{2n-1}\right) = \frac{1}{2} \ln 2 \,. \tag{8}$$

Next, we will use Leibniz's Formula (1), which we can write in the form

$$\lim_{n\to\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-1}\right) = \frac{\pi}{4} \,. \tag{9}$$

If we add the limits from (5) and (9), and divide both sides by 2, then we find

$$\lim_{n\to\infty} \left(1 + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{2n-3} - \frac{1}{4} \ln n\right) = \frac{\pi}{8} + \frac{1}{2}\left(\ln 2 + \frac{1}{2}\gamma\right) \,.$$

Subtracting $1/2$ times the limit in (5) from this equation gives

$$\begin{aligned}&\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-3}-\frac{1}{2n-2}\right)\\&-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+3}+\cdots+\frac{1}{2n-1}\right)\\&=\frac{\pi}{8}\,.\end{aligned}$$

Adding $1/2$ times equation (8) to both sides of this expression yields

$$\lim_{n\to\infty}\left(1-\frac{1}{2}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2n-3}-\frac{1}{2n-2}\right)=\frac{\pi}{8}+\frac{1}{4}\ln 2,$$

which can be rewritten as

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}\right)=\sum_{n=1}^{\infty}\frac{1}{(4n-2)(4n-3)}=\frac{\pi}{8}+\frac{1}{4}\ln 2\,. \tag{10}$$

Subtracting equation (3) from (10) yields

$$\sum_{n=1}^{\infty}\left(\frac{1}{(4n-2)(4n-3)}-\frac{1}{4n(4n-2)}\right)=\sum_{n=1}^{\infty}\frac{3}{4n(4n-2)(4n-3)}=\frac{\pi}{8}\,.$$

Finally, multiplication by $8/3$ gives the formula in the theorem. □

Remark 2. *Multiplying both sides of Theorem 1 by* 3 *produces a summation formula whose* $n=1$ *term is* 3*, and whose remaining terms give a formula for the fractional part of* π*:*

$$3\sum_{n=2}^{\infty}\frac{1}{n(2n-1)(4n-3)}=0.14159265\ldots\,.$$

3 Further identities

During the writing of this paper, the author also found a large number of related summation identities by combining Theorem 1 with other equations in the proof above, along with well-known zeta function identities of Euler (see [2]), such as

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\,,$$

and other classical summation identities found in [3]. Below, there is a selection of these identities given without proof, loosely organized by number of factors in the denominators of the summands. The first identity follows from the telescoping series

$$(1-1/2)+(1/2-1/3)+(1/3-1/4)+\cdots\,,$$

the second and third identities follow by rewriting (2) and (9), respectively, and the rest arise from liberal use of partial fractions and recursion relations.

The interested reader might like to prove the following results:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)} = 2\ln 2 \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n-3)} = \frac{\pi}{8} \tag{13}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-3)} = \frac{\pi + 2\ln 2}{4} \tag{14}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-3)} = \frac{\pi + 6\ln 2}{6} \tag{15}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-1)} = \frac{6\ln 2 - \pi}{2} \tag{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(4n-1)(4n-3)} = \frac{\ln 2}{2} \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)} = \frac{2\ln 2 - 1}{2} \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-1)(4n-3)} = \frac{\pi - 2}{16} \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)} = \frac{24\ln 2 - \pi^2}{6} \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(4n-1)} = \frac{72\ln 2 - 12\pi - \pi^2}{6} \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \frac{12 - \pi^2}{6} \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = \frac{\pi^2 + 24\ln 2 - 24}{6} \tag{23}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{\pi^2 - 6}{6} \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2 - 9}{3} \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)(4n+1)(4n-1)} = \frac{\pi - 3}{6} \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n-1)(8n+1)(8n-1)} = \frac{\pi(1+2\sqrt{2}) - 12}{24} \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n-1)(4n-1)(4n-3)} = \frac{6\ln 2 - \pi}{3} \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)(2n+1)} = \frac{\pi^2 - 12\ln 2}{6} \tag{29}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-3)} = \frac{\pi^2 + 8\pi - 24\ln 2}{18} \tag{30}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-1)} = \frac{\pi^2 + 24\pi - 120\ln 2}{6} \tag{31}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n-1)(4n-1)(4n-3)} = \frac{168\ln 2 - 32\pi - \pi^2}{18} \tag{32}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(6n+1)(6n-1)} = \frac{52 - 32\ln 2 - 27\ln 3}{16} \tag{33}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(3n+1)(3n-1)} = \frac{19 + 16\ln 2 - 27\ln 3}{10} \tag{34}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} = 10 - \pi^2 \tag{35}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)(4n+1)(4n-1)(8n+1)(8n-1)} = \frac{45 - \pi(3+8\sqrt{2})}{6} \tag{36}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)(2n-1)(3n+1)(3n-1)(6n+1)(6n-1)} = \frac{64\ln 2 + 27\ln 3 - 74}{20}. \tag{37}$$

Further identities like these can be discovered. For instance, since

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-3)} - \sum_{n=1}^{\infty} \frac{1}{n(4n-1)} = \sum_{n=1}^{\infty} \frac{(4n-1)-(4n-3)}{n(4n-1)(4n-3)} = 2\sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n-3)},$$

the reader could use the right-hand sides of (15) and (16) to find the value of

$$\sum_{n=1}^{\infty} \frac{1}{n(4n-1)(4n-3)}.$$

One might apply similar steps to other equations above, to find other new identities.

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References

[1] D. Blatner, *The Joy of* π, Walker and Company, 1997.

[2] W. Dunham, *Euler: The Master of Us All*, The Dolciani Mathematical Expositions **22**, Mathematical Association of America, 1999.

[3] L. B. W. Jolley, ed., *Summation of Series*, Courier Corporation, 2012.