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Simplifying difficult problems using dimensional analysis

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1 Introduction

"In Physics there is no place for muddled thinking. Those who really understand the nature of a given phenomenon must obtain the basic laws from dimensional considerations."

–Enrico Fermi

Dimensional analysis is a study of relationships between physical quantities expressed in terms of their fundamental units. The seven fundamental units include meters, seconds, moles, candelas, kelvin, ampere, and kilograms. The fundamental dimensions for these units are as follows: length, time, amount of substance, luminous intensity, temperature, electric current, and mass. Physical laws do not vary with respect to arbitrarily selected fundamental units of measurement. Take for example the equation $F = ma$. As mentioned by Jean-Baptiste Joseph Fourier in 1822 [1], the equation is independent of the units chosen for F, m and a. This is to say that, regardless of which units are chosen for F, m, and a, force will always be equal to mass times acceleration due to gravity. Mass could be in kg or in grams; acceleration can be in $\mathrm{m/s^2}$ or N/kg; and yet, the equation will not change as F will still be equal to ma. This is often referred to as dimensional homogeneity, which is the idea that the dimensions are the same on both sides of an equation, regardless of the units used. Dimensional analysis is the exploitation of this homogeneity. In history, numerous authors, such as Osborne Reynolds (1842–1912), James Clerk Maxwell (1831–1879) and Edgar Buckingham (1867–1940), were all associated with, and contributed to the advancement of dimension analysis [2]. Ain A. Sonin explains how dimensional analysis can help designing experiments by reducing the number of variables in a problem [3]. John Maynard Smith applied principles of dimensional analysis in mathematical biology [4]. Historic evidence points to the fact that several great applied mathematicians gained crucial insights into difficult problems with dimensional analysis.

In this expository article, we show that dimensional analysis can be a powerful yet simple method for solving difficult problems. Our aim is to promote the use of dimensional analysis in problem solving and attempt to clear the air of any mist surrounding dimensional analysis.

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2 Two Problems

Consider the following two well-known problems in elementary physics. We compare the classical text book methodology to solve these problems with the approach and conclusions of dimensional analysis. We hope that you might be pleasantly surprised by the simple effectiveness of the latter.

Problem 1: Given the length ℓ and mass m of a simple pendulum, find the formula for the time period T of a simple pendulum; see Figure 1.

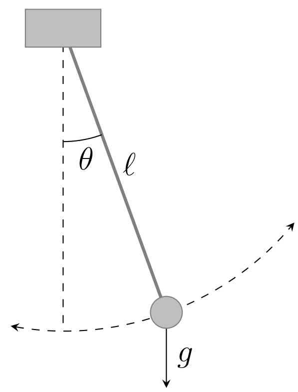

Figure 1: Finding the formula for time period of a pendulum.

Classical solution	**Dimensional analysis solution**
Newton's Laws + ordinary differential equation *Solution*: $\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}}$	Dimensional analysis principles + armchair experiment *Conclusion*: $\boxed{\frac{T}{\sqrt{\frac{\ell}{g}}} = C}$ where C is found by experiment.

Problem 2: Given the length ℓ and diameter D of a cylinder and a fluid of density ρ and viscosity μ flowing over the cylinder at velocity v, find the drag force F on the cylinder by the flowing fluid; see Figure 2.

Figure 2: Drag force per unit length [6]

Classical solution

Classical calculations are possible for inviscid flow. For viscous fluids, the flow patterns are tough to determine even for laminar flows, and very tough for turbulent flows.

Solution: ?? None in closed form.

Dimensional analysis solution

Dimensional analysis principles
+ armchair experiment

Conclusion:

$$\boxed{\frac{F}{\ell \rho v^2 D} = f\Big(\frac{vD\rho}{\mu}\Big)}$$

where f is found by experiment.

We hope that the conclusions of dimensional analysis will appear startling and like a "*rabbit out of the hat*" breakthrough to the problem. In both cases, we can proceed further via efficient experimentation to give accurate practical solutions. How does dimensional analysis work out this sleight of hand magic? We proceed to explain the science and mathematics of this trick to the uninitiated. A detailed discussion of these two examples will suffice.

2.1 Solving Problem 1 by dimensional analysis

The formula for the dependence of the time period on the length of the pendulum has been a classical problem for generations. The formula $T = 2\pi\sqrt{\frac{\ell}{g}}$ is derived using Newton's laws and solving a differential equation. What if we were to show you a way to derive this formula without any complex mathematics and just an armchair experiment? This is where dimensional analysis comes in. The steps of dimensional analysis are as follows.

Algorithm to exploit dimensional analysis

Step 1: Guess the variables that affect the time period.

This step requires some inspiration as it requires one to think carefully about the possible variables that can affect the time taken for one oscillation. This is the step that could make or break the analysis. If the final conclusion of dimensional analysis is incorrect, then one can be sure that this first step was the single source of the error. Here, we guess that the variables that affect the time period T of the pendulum are the length ℓ of the pendulum, the mass m of the bob, and the acceleration g due to gravity. Our aim is to find the function f below:

$$T = f(\ell, m, g)\,. \tag{1}$$

Step 2: Recast the above equation to make the left-hand side dimensionless.

The left-hand side of (1) must be made dimensionless using variables from the function on the right-hand side of (1), namely ℓ, m, and g. Divide both sides by $\sqrt{\ell/g}$ which, like T, has seconds as units, to make the left-hand side dimensionless:

$$\frac{T}{\sqrt{\ell/g}} = f^*(\ell, m, g) \qquad \text{where} \qquad f^*(\ell, m, g) = \frac{f(\ell, m, g)}{\sqrt{\ell/g}}\,.$$

Since the left-hand side is dimensionless, then so is the function $f^*(\ell, m, g)$.

Step 3: Apply the principles of dimensional analysis via armchair experiments.

Each such experiment will reduce the number of variables by one. For the first experiment, imagine that the function f^* is known to a group of mathematicians. All of the mathematicians use the same units for length ℓ and time t but some mathematicians use the unit of mass as kg and the rest in grams. The mathematicians are told to calculate

$$T^* := \frac{T}{\sqrt{\ell/g}} = f^*(\ell, m, g)\,.$$

Now all of the mathematicians must get the same number for T^* since it is dimensionless and therefore cannot depend on the particular unit used for the mass m. The calculated values for f^* therefore also don't depend on the unit used for the mass. Since different units are used by the mathematicians, the function f^* cannot depend on m. We can therefore write

$$T^* = \frac{T}{\sqrt{\ell/g}} = f^*(\ell, g)\,. \tag{2}$$

Thus, this armchair experiment uses the principle of dimensional analysis to eliminate mass as one of the variables of f^*.

Now, imagine that all of the mathematicians use the same units for length ℓ but some mathematicians use different units for time t. As above, this will give varying values of g for the two groups of mathematicians but the value of T^* calculated by the group will be the same for all of the mathematicians, since T^* is dimensionless. We conclude that f^* cannot depend on g and it follows that

$$T^* = f^*(\ell)\,.$$

Thus, the experiment uses dimensional analysis to eliminate g as a variable of f^*.

Finally, imagine that some of the mathematicians are told to take the unit of length as meters and the rest as cm. This will give varying values of the length ℓ of the pendulum. Since all of the mathematicians arrive at the same value of T^* and thus of $f^*(\ell)$, it follows that $f^*(\ell)$ does not depend upon the length ℓ. This means that

$$T^* = \frac{T}{\sqrt{\ell/g}} = c$$

for some (yet unknown) constant; that is,

$$T = c\sqrt{\frac{\ell}{g}}\,. \tag{3}$$

Step 4: Perform a (real) experiment to find the value of c in the expression above.

All we did to arrive at our solution to Problem 1 was to use the principle of dimensional analysis and determine the consequence of a sequence of armchair experiments. This led to a spectacular simplification of the problem.

Please note that the sequence of armchair experiments was carried out in a way such that only one independent variable varied in each experiment. For a more detailed discussion, see Barenblatt's book [6].

Numerical example

Question: What is the ratio of the time period of a 1m long pendulum on the moon to that on the earth?

Solution:

$$\frac{T_{moon}}{T_{earth}} = \frac{\sqrt{\frac{\ell}{g_{moon}}}}{\sqrt{\frac{\ell}{g_{earth}}}} = \sqrt{\frac{g_{earth}}{g_{moon}}} = \sqrt{\frac{9.81}{1.63}} \sim 2.5\,.$$

2.2 Solving Problem 2 by dimensional analysis

Problem 2 is significantly more difficult than Problem 1 and the dimensional analysis will now require an additional clever manipulation of the variables as compared with the analysis in Problem 1. The main purpose of this section is to explain this additional manipulation of the independent variables.

Algorithm to exploit dimensional analysis

Step 1: Guess the variables that affect the drag force per unit height.

This step is the same as the first step for solving Problem 1. We guess that the variables that affect the drag force per unit height $\frac{F}{\ell}$ are the velocity v of the cylinder moving in the medium, the diameter D of the cylinder, the density ρ of the fluid, and the viscosity μ of the fluid. That is, we guess the independent variables determine $\frac{F}{\ell}$ as some function $f(v, D, \rho, \mu)$:

$$\frac{F}{\ell} = f(v, D, \rho, \mu)\,. \tag{4}$$

Step 2: Make the left side of (4) dimensionless by using variables from the right side.

In order to do this, divide both sides of the equation by $v^2 D\rho$:

$$\Pi := \frac{F}{\ell v^2 D\rho} = \frac{f(v, D, \rho, \mu)}{v^2 D\rho}\,.$$

It is easy to verify that the left side of the above equation, here called Π, is dimensionless. The right side of the equation can thus be expressed as some function $f^*(v, D, \rho, \mu)$.

Step 3: Introduce the variable $Re = \frac{vD\rho}{\mu}$.

This step is the new clever manipulation. Note that Re is dimensionless. It is known in literature as the *Reynolds number*. Since $\mu = vD\rho/Re$, the viscosity depends on the variables v, D, ρ and Re. We can therefore view $f^*(v, D, \rho, \mu)$ and thus $f^*(v, D, \rho, \mu)/(v^2 D\rho)$ as a function f^{**} of those four variables:

$$\Pi = \frac{f(v, D, \rho, \mu)}{v^2 D\rho} = f^{**}(v, D, \rho, Re)\,. \tag{5}$$

Our objective in this problem is to find f^{**}.

Step 4: Perform armchair experiments exactly as in Problem 1.

The expression (5) is used by a group of mathematicians, all of whom use the same unit of length and time but a different unit for mass. Some mathematicians are told to use kg as the unit of mass and the others are told to use grams. The value of ρ thereby used by the mathematicians will thereby differ. In contrast, the entire group will get the same answer for Π because it is dimensionless. Therefore, all of the mathematicians will calculate the same value for $f^{**}(v, D, \rho, Re)$ even though their values of ρ differ. We conclude that the function f^{**} does not depend on ρ. We express this as follows:

$$\Pi = f^{**}(v, D, Re)\,.$$

Now, some mathematicians are told to take the unit of time as seconds and others as minutes. This will give varying values of v but constant values of D and Re. Since the calculated values of $\Pi = f^{**}(v, D, Re)$ are also constant, we conclude that f^{**} does not depend on v, and we can write

$$\Pi = f^{**}(D, Re)\,.$$

Finally, some mathematicians are told to use meters as the unit of length and others are told to use cm. This will give varying values of D, the diameter of the cylinder, but the calculated values of $\Pi = f^{**}(D, Re)$ will still all be the same. Therefore, $f^{**}(D, Re)$ does not depend on D:

$$\Pi = \frac{F}{\ell v^2 D\rho} = f^{**}(Re)\,.$$

Step 5: Perform (real) experiments to determine f^{**} experimentally.

The experiments have a single variable (Re), reduced from four variables by dimensional analysis. Assuming that one would take 10 experiments per variable of the function f^{**} for calculating the value of Π, the number of experiments have reduced from around 10^4 to just $10^1 = 10$ experiments.

Let us pause and highlight the power of dimensional analysis. It is truly remarkable. We have been able to solve a problem which could not have been solved by normal calculations. We can determine the dependence of Π on a single variable Re through experiments and present these results as a graph. With this graph, we can determine the drag force of a cylinder of any diameter, any length and flowing through any fluid with any velocity. The results are reliable for turbulent flow too. The graph for $\Pi = f^{**}$ has been obtained by experimentalists; for instance, see the graph in Figure 3 below, reproduced from [7]. It shows Π as a function of Re.

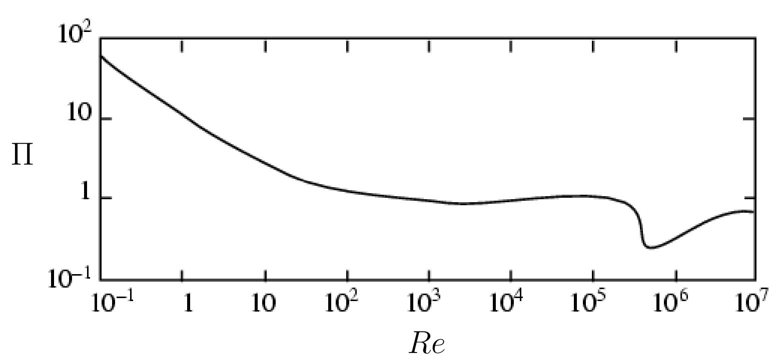

Figure 3: Π as a function of Re [7]

Numerical example

We illustrate the use of the above graph with a numerical example.

Question: Find the drag force on a cylinder of length $\ell = 2$m and diameter $D = 0.2$m by air flowing over it with velocity $v = 12$m/s.
Data:

$$\begin{aligned} \ell &= 2\text{m} \\ D &= 0.2\text{m} \\ \rho &= 1.2\text{kg/m}^3 \\ \mu &= 1.81 \times 10^{-5}\text{Pa/s} \\ v &= 12\text{m/s} \end{aligned}$$

Solution:
Step 1: Calculate

$$Re = \frac{vD\rho}{\mu} = \frac{12 \times 0.2 \times 1.2}{1.81 \times 10^{-5}} \sim 1.7 \times 10^5 \, .$$

Step 2: Use the value calculated above, refer to the graph, and find that $\Pi \sim 1$. We can now use this value of Π to calculate the drag force. From the above section, we determined that Π is equal to $\frac{F}{\ell v^2 D\rho}$. Therefore,

$$F = \ell v^2 D\rho\Pi = 2 \times 12^2 \times 0.2 \times 1.2 \times 1\,\text{N} \sim 69\,\text{N} \, .$$

3 Similarity and scaling

It is very useful in design to build and study smaller prototypes of large scale objects. It is even more useful if experimental studies on these prototypes can be used to predict the results for the larger objects accurately. This is possible through the idea of similarity and scaling. This idea follows easily from dimensional analysis. Two objects are called similar if they differ only in dimensional quantities but the associated dimensionless parameters are identical.

Consider the following equation from the previous section:

$$\Pi = \frac{F}{\ell v^2 D\rho} = f^{**}(Re) \, .$$

Here, both Π and Re are dimensionless and Π is a function of Re as shown in Figure 3.

Consider a scenario where we are investigating the drag force of a very corrosive and harmful industrial fluid on a cylinder. We are not keen on conducting such experiments in a laboratory. Instead, we use the idea of similarity. We focus on the Reynolds number of the desired experimental study and calculate the required value of $\frac{vD\rho}{\mu}$. We prefer working with a more benign fluid and so we change ρ and μ. But, we can still design the experiment to obtain the same Reynolds number by appropriately changing the values of v and D in our prototype experiment. The results of our experiment can easily be used for the actual model because we have ensured the same value of Re, the independent variable. We determine the value of Π with our laboratory experiment:

$$\Pi = \frac{F}{\ell v^2 D \rho} = f^{**}(Re)\,.$$

The value of Π of the actual design is then known and we can proceed to calculate the force per unit height accurately.

For a more intricate model we can have equations of the form

$$\Pi = f_1^{**}(\Pi_1, \Pi_2, \Pi_3)\,.$$

To obtain reliable values for the dependent variables, Π, we design experiments to obtain the same values of the independent variables Π_1, Π_2, Π_3, as required in the actual design. Thus, dimensional analysis enables scaling through the simple idea of similar dimensionless variables.

Numerical example

Question: Suppose that we investigate the flow of corrosive mercury over a cylinder. Can we instead perform the experiments with water and predict the results for the flow of mercury?

Data:

$$\begin{aligned} \nu_{\text{mercury}} &:= \frac{\mu_{\text{mercury}}}{\rho_{\text{mercury}}} = 1 \times 10^7 \\ \nu_{\text{water}} &:= \frac{\mu_{\text{water}}}{\rho_{\text{water}}} = 1 \times 10^6\,. \end{aligned}$$

Solution: Our objective can be easily achieved if our value of Π_1 is same for the design requirement as for the prototype. We propose to use water for the experiment; then

$$\nu_{\text{water}} D = 10\,\nu_{\text{mercury}} D\,.$$

This ensures that the value of Re is the same for the requirement and the prototype.

4 Limitations of dimensional analysis

We would like to point out that in both dimensional analysis solutions, the final step was experimentation. This is clearly a limitation of dimensional analysis as we can only get the complete solution of the problem by performing a (real) experiment. Also, the whole analysis can be flawed if the first guess of identifying the variables is incorrect. Making accurate guesses for the reliable dependent variables is an art, so dimensional analysis is a powerful tool made even more so by the experts.

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