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The last problem section

Yagub N. Aliyev[1]

1 Introduction

There used to be journal by the name *Publications of the Faculty Electrical Engineering - Series Mathematics* (*Publikacije Elektrotehničkog Fakulteta - Serija Matematika*) which had a problem section, popular among the problem-solving community. In 2007, it rebranded itself into a fully-research journal, *Applicable Analysis and Discrete Mathematics* with no problem section anymore. The last problem section contained 9 new problems, including one proposed by me, whose solution was never published. In this short paper, we will try to do justice for the problem proposed by me, by discussing its origins, relation to other topics in mathematics, and presenting the solution after 15 years since its publication.

Problem. [2] *Do irrational numbers* $a, b > 1$ *exist so that* $|a^n - b^m| > 2006$ *for all integers* $m, n > 1$*?*

This problem was inspired by a problem proposed by V. Senderov and A. Spivak in The 23rd Tournament of Towns in 2002 for 10th-11th graders:

Problem. [12] *Do irrational numbers* $a,\ b > 1$ *exist so that* $\lfloor a^n \rfloor \neq \lfloor b^m \rfloor$ *for all positive integers* m, n*?*

The answer to both of these questions is affirmative. It is obvious that if $|a^n - b^m| > 1$, then $\lfloor a^n \rfloor \neq \lfloor b^m \rfloor$. Therefore, it is sufficient to solve the first problem only.

2 Solution

Let $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$. Then

$$\left|\alpha^n - \beta^m\right| = \left|2^{\frac{\sqrt{2}n}{2}} - 2^{\frac{m}{2}}\right|.$$

If $n > m$, then

$$\left|2^{\frac{\sqrt{2}n}{2}} - 2^{\frac{m}{2}}\right| = 2^{\frac{\sqrt{2}n}{2}} - 2^{\frac{m}{2}} > 2^{\frac{\sqrt{2}m}{2}} - 2^{\frac{m}{2}} = 2^m \left(2^{\frac{(\sqrt{2}-1)m}{2}} - 1\right).$$

Both of the factors 2^m and $2^{\frac{(\sqrt{2}-1)m}{2}} - 1$ grow infinitely large as m tends to infinity.

[1]Yagub N. Aliyev is Assistant Professor at ADA University, Baku, Azerbaijan.

If $n \le m$, then we shall use the following well-known inequality [9, p. 133]:

$$\left|\sqrt{2} - \frac{m}{n}\right| > \frac{1}{3n^2}\,. \tag{1}$$

By (1),

$$\left|2^{\frac{\sqrt{2}n}{2}} - 2^{\frac{m}{2}}\right| = 2^{\frac{m}{2}}\left|2^{\frac{n}{2}(\sqrt{2}-\frac{m}{n})} - 1\right|$$
$$> 2^{\frac{m}{2}}\left|2^{\frac{n}{2}(\pm\frac{1}{3n^2})} - 1\right| = 2^{\frac{m}{2}}\left|2^{\pm\frac{1}{6n}} - 1\right| \ge 2^{\frac{n}{2}}\left|2^{\pm\frac{1}{6n}} - 1\right| = \pm 2^{\frac{n}{2}}\left(2^{\pm\frac{1}{6n}} - 1\right).$$

We shall estimate the last term using l'Hospitale's rule:

$$\pm\lim_{x\to\infty}\frac{2^{\pm\frac{1}{6x}} - 1}{2^{-\frac{x}{2}}} = \pm\lim_{x\to\infty}\frac{2^{\pm\frac{1}{6x}}\left(\mp\frac{1}{6x^2}\right)\ln 2}{-2^{-\frac{x}{2}}\frac{\ln 2}{2}} = \lim_{x\to\infty}\frac{2^{\frac{x}{2}\pm\frac{1}{6x}}}{3x^2} = \infty\,.$$

Then there exist sufficiently large integers M, N for which

$$\left|\alpha^n - \beta^m\right| > 2006$$

for all $m \ge M$ and $n \ge N$. By taking $a = \alpha^N$ and $b = \beta^M$, we obtain that the inequality $|a^n - b^m| > 2006$ is true for all positive integers n, m.

It remains only to choose M and N so that a and b are both irrational numbers. Let us choose M as odd number and N as even number. Then $a = \alpha^N$ and $b = \beta^M$ are of the forms $2^{k\sqrt{2}}$ and $2^{\ell}\sqrt{2}$, respectively, for positive integers k, ℓ. The number $2^{\ell}\sqrt{2}$ is obviously irrational. The other number $2^{k\sqrt{2}}$ is irrational because $2^{\sqrt{2}}$ is a transcendental number (see below) and any positive integer power of a transcendental number is again transcendental and therefore irrational. The transcendency of the number $2^{\sqrt{2}}$ was first proved approximately 90 years ago by R. Kuzmin [8] and C. Siegel [11].

This completes the solution. □

3 Remarks

A real number is *algebraic of kth degree* if it is a root of a polynomial of kth degree with rational coefficients and it is not a root of any such polynomial of smaller degree. A number is *transcendental* if it is not an algebraic number; i.e., if it is not a root of any polynomial with rational coefficients. The existence of transcendental numbers was first proved by J. Liouville in 1844. His proof was based on the folowing inequality [3, Theorem 270]:

If γ is an algebraic number of kth degree, then for arbitrary coprime integers p, q ($\gamma \ne p/q$) then following inequality is true:

$$\left|\gamma - \frac{p}{q}\right| > \frac{1}{Aq^k}, \tag{2}$$

where $A > 0$ is a constant. For an elementary proof of this inequality, see [4]. The inequality (1), which was used above in the proof, is a special case of Liouvilles inequality (2).

For example, let $\ell > 1$; then the number

$$\omega = 1 + \frac{1}{\ell} + \frac{1}{\ell^2} + \cdots + \frac{1}{\ell^{n!}} + \cdots$$

does not satisfy (2) and is therefore a transcendental number (see [5, p. 17]).

In 1900, at The International Congress of Mathematicians, which was held in Paris, France from 6 August to 12 August, D. Hilbert stated 23 unsolved problems of the XIX century. The seventh problem was on the transcendency of the numbers β^α, where $\beta \neq 0, 1$ is an algebraic number and α is an irrational algebraic number (see [1]). In particular, the number $2^{\sqrt{2}}$ was noted by him. In 1934, A. Gelfond [5, p. 48-56] and T. Schneider [10] completely solved this problem. Later in this direction, A. Gelfond obtained the following inequality (see [5, p. 84] or [6]):

If $|\alpha|, |\beta| \neq 0, 1$; $\ln\alpha / \ln\beta$ is irrational; and $|\beta|^m \geq |\alpha|^n$, then

$$\left|\alpha^n - \beta^m\right| > |\beta|^m e^{-\ln^{3+\varepsilon}|m|},$$

where $\varepsilon > 0$.

It was noted in [5, p. 84] that this inequality validates the claim that two geometric series with well-chosen algebraic factors greater than 1 has the property that the differences of their terms are infinitely large for sufficiently large values of powers. In other words, if $\alpha, \beta > 1$ are well-chosen algebraic numbers, then

$$\lim_{n,m\to\infty} \left|\alpha^n - \beta^m\right| = \infty, \tag{3}$$

where n, m are integers. This can be used to give another solution for the problems above.

It is possible to prove the following generalizations using the same method.

Theorem 1.
Let $M > 0$ be arbitrary real number. Then there exist a transcendental number $\alpha > 1$ and an algebraic number $\beta > 1$ such that

$$\left|\alpha^n - \beta^m\right| > M,$$

for all positive integers n, m.

Theorem 2.
Let $\beta > 1$ be an algebraic number and let $\gamma > 1$ be an algebraic number of degree $k > 1$. Then

$$\lim_{n,m\to\infty} \left|\alpha^n - \beta^m\right| = \infty,$$

where $\alpha = \beta^\gamma$, and n, m are integers.

For the proof, instead of (1), we can use Liouville's inequality (2) and the remaining part of the proof is similar. It is also possible to chose both of the numbers $\alpha, \beta > 1$ in (3) to be transcendental numbers. For example, we can take the numbers $2^{\sqrt{6}}$ and $2^{\sqrt{3}}$.

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