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The beauty of the Golden Ratio

Thomas Britz[1]

Introduction

The Golden Ratio in sunflowers, sort of

The Golden Ratio φ is one of the numbers most revered by non-mathematicians. According to widespread belief, the Golden Ratio appears prevalently in nature. Many people also believe that φ determines beauty in art, architecture and faces, either due to psychology or more mystical means. As I explained in a recent article [3], these beliefs are false. The Golden Ratio does sometimes appear in nature, such as in some spiralling patterns of some nuts and flowers. However, this is approximate and relatively rare. As for the claim that the Golden Ratio determines Beauty in art and faces, there is no evidence for this, and much evidence against it.

Nevertheless, it is true that φ is both prevalent and beauty-inducing, just not in nature or art. The Golden Ratio appears surprisingly often, in surprisingly beautiful ways – in maths. This note showcases a few of these surprising and beautiful ways.

The Pythagoreans and the Golden Ratio

As far as we know, the Golden Ratio, or *Divine Proportion* as it is also sometimes called, was first discovered by the Pythagoreans, roughly 2400 years ago. They were a cult of mathematicians who discovered many mathematical truths, including Pythagoras' famous theorem. Some of these truths arose from empirical observations involving natural numbers and their ratios, such as musical harmonics appearing in ratios 1:2 or 2:3. Other truths were derived conceptually, especially via geometry. To the Pythagoreans, the ubiquity of whole numbers and whole number ratios in nature suggested that these had numerological, philosophical and even ethical significance. As the symbol of their cult, they chose the pentagram which to them, with its five-fold symmetries, symbolised balance and health.

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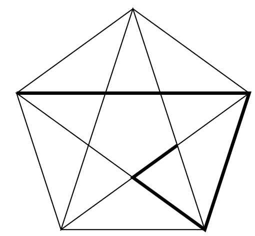
The pentagram is mathematically fascinating, not least since a curious ratio keeps appearing when you study this symbol. In the pentagram to the right, the four thick black line segments grow in length by a fixed ratio φ at each step. For instance, the long thick horizontal line is φ longer than the side length of the pentagram. This constant φ is indeed the Golden Ratio



$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\,.$$[2]

This is an irrational number, not a ratio of natural numbers. When the Pythagorean Hippasus discovered irrational numbers, the Pythagoreans suffered a crisis of reality[3]. They believed that you can only divide matter and magnitude a finite number of times until you get *atoms*. These are the indivisible units of reality that all existence is made out of. If one line length consists of p "line length atoms" and another line length consists of q such atoms, then the ratio between these two lengths is p/q, a rational number. Irrational numbers such as $\sqrt{2}$ and φ proved that not everything is built of atoms; some things are infinitely divisible. To the Pythagoreans, this realisation pulled a rug of certainty from under their feet, revealing below an abyss of infinity.

Surprising and beautiful instances of the Golden Ratio

To be honest, I would not blame you, dear Reader, if the expression $(1+\sqrt{5})/2$ left you indifferent. It does not look particularly interesting or nice to me either. However, it is possible to express φ in other ways, some of which you might find surprising or beautiful. For instance, you can quickly check that the Golden Ratio φ is the positive solution to the simple equation

$$x^2 = x + 1\,.$$

Therefore, the Golden Ratio can be expressed as the infinitely nested square roots

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}\,.$$

It can also be expressed as the continuing fraction

$$\varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}\,.$$

Remarkably, its inverse, roughly 0.618, can be expressed in nearly the same way:

$$\frac{1}{\varphi} = \varphi - 1 = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}\,.$$

[2]Can you prove this?

[3]and, according to your legend of choice, they either killed or exiled poor Hippasus.

The Golden Ratio and Fibonacci numbers

The Golden Ratio φ is intimated related to other famous and popular numbers, the Fibonacci numbers. As you might already know, these are the numbers

$$0\,,1\,,1\,,2\,,3\,,5\,,8\,,13\,,21\,,\ldots$$

that you get by starting with 0 and 1 and adding consecutive pairs of numbers to get the next; for instance, $0+1=1$; $1+1=2$; $1+2=3$; $2+3=5$; and so on. The Golden Ratio is the speed at which these numbers eventually grow. More precisely, the ratio between any two consecutive Fibonacci numbers F_{n-1} and F_n tends towards φ as n grows large: For instance,

$$\begin{aligned}
F_2/F_1 &= \ \ 1/1 \ \ = 1\\
F_3/F_2 &= \ \ 2/1 \ \ = 2\\
F_4/F_3 &= \ \ 3/2 \ \ = 1.5\\
F_5/F_4 &= \ \ 5/3 \ \ \approx 1.667\\
F_6/F_5 &= \ \ 8/5 \ \ = 1.6\\
F_7/F_6 &= 13/8 \ \ = 1.625\\
F_8/F_7 &= 21/13 \approx 1.615
\end{aligned}$$

and so on. In other words, the Golden Ratio φ is determined by the Fibonacci numbers via the identity

$$\varphi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}\,.$$

The converse is also true: Fibonacci numbers are also determined by φ, by an identity called *Binet's Formula* [5]:

$$F_n = \frac{1}{\sqrt{5}}\big(\varphi^n - (-\varphi)^{-n}\big)\,.$$

Given that each Fibonacci number F_n is a whole number, it is beautifully surprising that it can be expressed like this in terms of the irrational Golden Ratio φ like this. Indeed, since the term $(-\varphi)^{-n}$ quickly grows very small, we can express all Fibonacci numbers F_n for all n simply as

$$F_n = \left[\frac{\varphi^n}{\sqrt{5}}\right]\,.$$

Here, the square brackets mean that $\frac{\varphi^n}{\sqrt{5}}$ is rounded to its nearest integer.

The Fibonacci numbers are determined by the recursive identity

$$F_n = F_{n-1} + F_{n-2}$$

and the powers of the Golden Ratio are determined similarly:

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}\,. \tag{1}$$

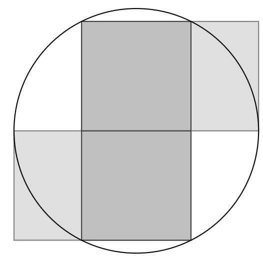
This identity follows by multiplying both sides of the equation $\varphi^2 = \varphi + 1$ by φ^{n-2}. Here, n can be any real number.

20 geometrical instances of the Golden Ratio

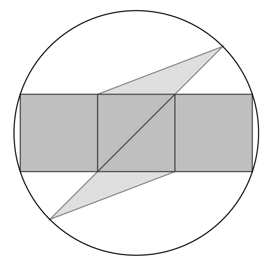
It is possibly in geometry that the Golden Ratio is most popular and celebrated. The Golden Ratio seems to hide nearly everywhere in geometry and the hunt for φ among geometric shapes is very popular. Maybe you would like to join this hunt too?

In this section, I present 20 of these geometric instances of the Golden Ratio but there are countless others. I present only the instances of φ but leave the underlying proofs and derivations for you, dear Reader, to fill in as a hopefully fun challenge.

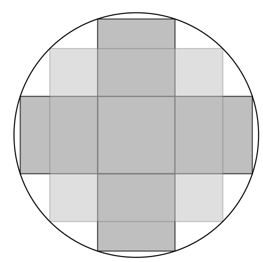
The first geometrical instance of the Golden Ratio follows easily from the algebraic expression for φ. Stack two squares of the same size on top of each other and place them inside a circle. Now extend each square horizontally to the edge of the circle to form two rectangles. The area of each rectangle is φ times larger than that of each square.



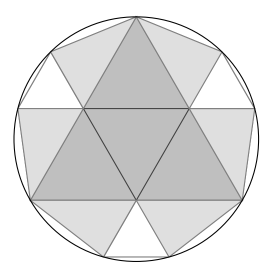
In the following picture by master Golden Ratio finder Tran Quang Hung [4], place three squares of the same size in a row inside a circle and draw a circle diameter through two opposite corners of the middle square. The resulting inner triangles are φ times larger in area than the outer triangles.



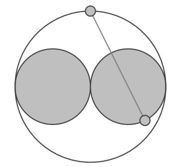
Here is another picture by Tran Quang Hung [4]. Draw a cross from five squares of the same size inside a circle and draw a bigger square inside the circle as well. The eight outer squares and rectangles have the same area which is φ times smaller than the area of each of the four inner rectangles. These areas are φ times smaller than that of the middle square.



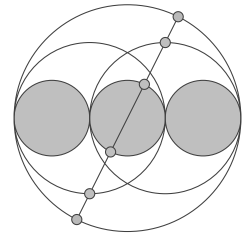
This picture extends a geometric figure by the professional artist and amateur mathematician George P. Odom Jr. [6, 9, 12]. Pack four equal-sided triangles of the same size into one dark-grey Triforce triangle inside a circle. The area of each dark-grey triangle is φ times larger than the area of each light-grey triangle which is φ times larger than the area of each white triangle.



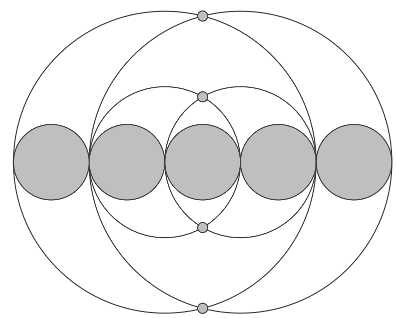
This simple figure is by Tran Quang Hung [4]. Place two same-sized circles next to each other and draw a circle around them. Draw a line through the top of the big circle and the right circle centre. The distance from the top of the big circle to the intersection point drawn is φ times the radius of the big circle.



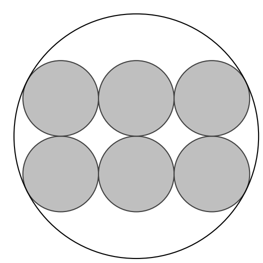
Tran Quang Hung's picture above is here expanded. This new picture is not as elegant or pretty but the intersection points between the line and the four circles give many φ ratios: using the shortest distance between these points as unit measure, the distances between the points include 1, φ, φ^2, φ^3, φ^4 and φ^5.



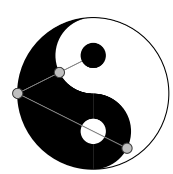
The following figure has been adapted from a short construction of φ by Kurt Hofstetter in 2002 [6]. Place five circles of the same size in a row and draw circles around them as drawn to the right. Then the distance from the bottom small intersection point to the top big intersection point is φ times the distance between the two small intersection points.



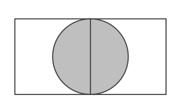
This simple and pretty figure is from [10]. Pack six circles of the same size into two rows of three, and place this grid of circles inside a bigger circle. The radius of the large circle is φ times larger than the diameter of the small circles.



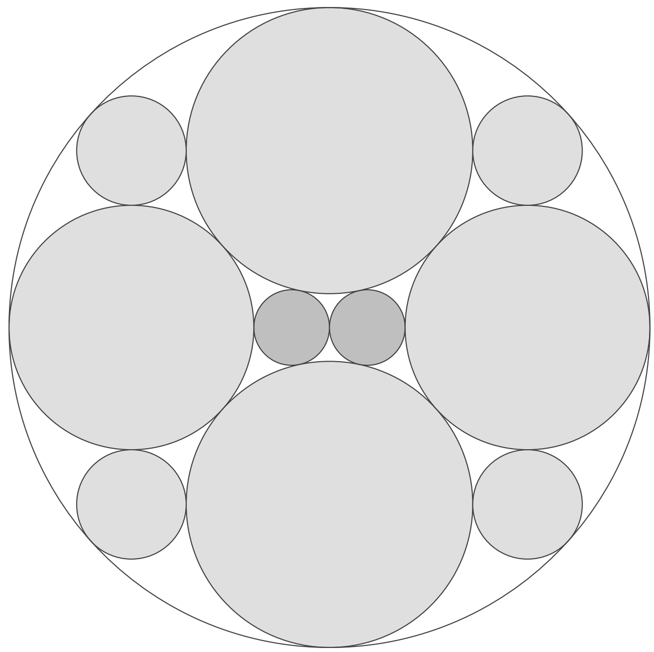
This Yin-Yang variation of Tran Quang Hung's elegant picture at the top of the page is by John Arioni [4]. Draw lines from the left-hand side of the symbol through the small circles. The distances from the left-hand side point to the other two points are respectively φ and φ^{-1} times the times radius of the symbol.



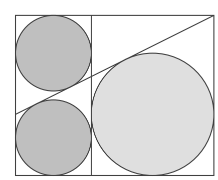
John Arioni [4] also gave an elegant picture, here simplified. The smallest distance from the upper-left corner to a point on the circle is φ times smaller than the circle diameter, whereas the largest distance from that corner to a point on the circle is φ times bigger than the diameter.



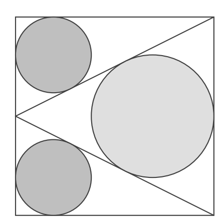
The elegant picture below is by Jerzy Kocik [7] who made it a window of his house. The radius of the encompassing circle is φ^2 larger than that of the left and right circles which in turn is φ times larger than the diameter of the inner circles which, in turn, is $\frac{\varphi}{\sqrt{5}}$ times smaller than that of the four corner circles. Many other φ ratios appear.



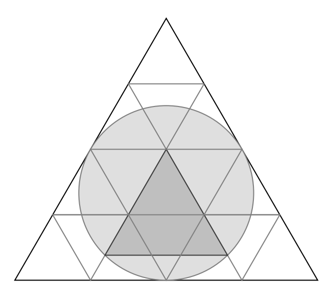
This figure is due to Kadir Altinas [4]. Fit two circles of the same size and a larger circle into a rectangle with dividing lines as shown. Then the bigger circle has φ times larger radius than the two smaller circles.



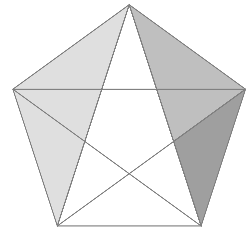
A nice variation of the above figure was found by Ercole Suppa [4]. Draw two lines from the midpoint of the left side of the square to the opposite diagonals, and fit circles into the three resulting areas. The big circle has φ times larger radius than the two smaller circles.



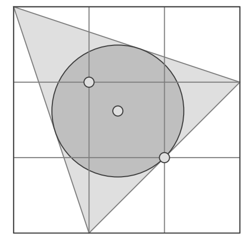
This figure is a slight adaptation of another figure due to Kadir Altinas [4]. The side length of the gray triangle is φ times larger than the side length of each of the small triangles.



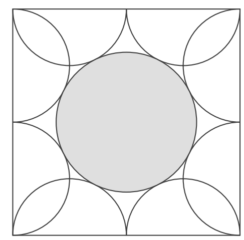
There are five sizes of triangles in the pentagram. The big white triangle has area φ larger than the grey triangle to the left. It in turn has area φ times larger than the darker grey triangle to the upper-right. That triangle has area φ times larger than the dark grey triangle to the right. It, finally, it has area φ times larger than each of the smallest triangles in the pentagram.



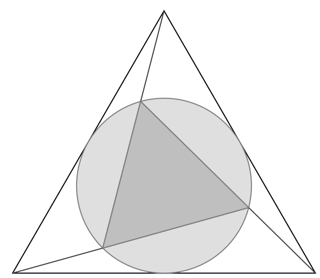
In this figure by Tran Quang Hung [4], the distance from the centre of the circle to the furthest corner of the middle square is φ times smaller than the distance from the circle centre to the closest corner.



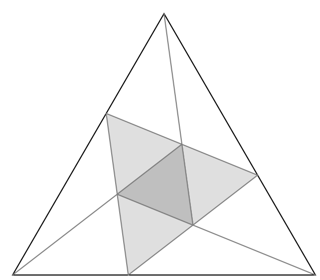
This pretty figure is from Problem 22 of the 2014 American Mathematics Competition 10B Problems [1]. The radius of the middle circle is φ times smaller than the diameter of the semi-circles.



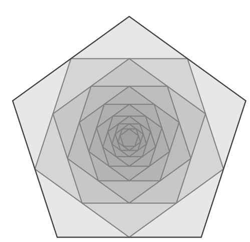
This is yet another figure by Tran Quang Hung [4]. Here, a circle is drawn inside an equal-sided triangle. Three lines are drawn from each corner to points on the circle in such a way that the lines touch and form a triangle within the circle. The length of each line is φ times larger the side length of the inner triangle.



Also due to Tran Quang Hung [4], here is a variation of the figure above. The three lines are drawn from each corner of a triangle in such a way that they end in the midpoints of the sides of an inner triangle. The area of each white triangle is φ times larger than the area of each of the gray triangles.



This picture is inspired by a figure by Tran Quang Hung [4]. As the pentagons shrink inwards, the side length of each pentagon is $\varphi/2$ times that of the preceding pentagon.



Further reading

In this article, I have presented just a few of the many and varied instances of φ in maths, particularly in geometry. Most of the geometrical figures were re-drawn, modified or inspired by figures from the extensive list [4] on the fantastic site *cut-the-knot.org* by the late Alexander Bogomolny. I have only very briefly addressed the rich and fascinating history of the Golden Ratio. The excellent book [8] provides a thorough and very readable account of this history. It also gives a clear account of how the popular urban myths around the Golden Ratio arose and how they persist.

For more fascinating infinite fraction expressions for φ, see the excellent article [11]. To learn more about some of the less-known properties of Fibonacci numbers, see [2].

References

[1] Problem 22, 2014 American Mathematics Competition 10B Problems, `https://artofproblemsolving.com/wiki/index.php/2014_AMC_10B_Problems/Problem_22`, last accessed on 25-08-2022.

[2] T. Britz, Fibonacci numbers: Brief history and counting problems, *Parabola* **56** (1) (2020), 6 pages.

[3] T. Britz, Does Amber Heard really have the world's most beautiful face? An expert explains why the Golden Ratio test is bogus, *The Conversation*, 25 July 2022.

[4] Cut the knot, *Golden Ratio in Geometry*, `https://www.cut-the-knot.org/do_you_know/GoldenRatio.shtml`, last accessed on 21-08-2022.

[5] Fibonacci numbers, *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org/A000045`, last accessed on 21-08-2022.

[6] K. Hofstetter, A simple construction of the Golden Section, *Forum Geometricorum* **2** (2002), 65–66.

[7] J. Kocik, Golden window, *Math Magazine* **83** (2010), 384–390.

[8] M. Livio, *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*, Broadway, 2002.

[9] G. Odom, Elementary Problem E3007, *The American Mathematical Monthly* **90** (1983), 482.

[10] C.A. Pickover, *A Passion for Mathematics*, John Wiley & Sons, 2005.

[11] R. Schneider, Fibonacci numbers and the golden ratio, *Parabola* **52 (3)** (2016).

[12] J. van de Craats, Elementary Problem E3007, solution, *The American Mathematical Monthly* **93** (1986), 572.