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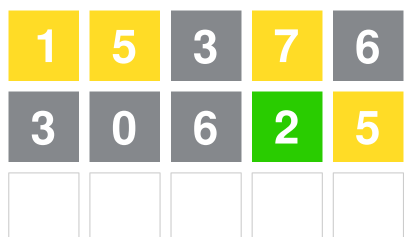
Problems 1681–1690

Parabola would like to thank a number of readers who contributed problems for this issue: Toyesh Prakash Sharma (Problem 1682 and the idea behind Problem 1686), Timothy Hume (Problem 1683), Soham Dutta (Problem 1685) and Jason Zimba (Problem 1689).

Q1681 The recently popular game "Wordle" challenges you to guess a secret five–letter word. You may enter any word from the official "Wordle" word list, and you will be given some information in return. You are allowed a maximum of six guesses.

In the not–at–all well–known game "Squardle", you have to guess a secret square number, and you may enter any five–digit square. Here is an example of the start of a game.



Two guesses have been entered so far: $124^2 = 15376$ and $175^2 = 30625$. When a digit is highlighted green in the diagram, it indicates that the digit occurs in the secret square in the same location as it is in the guess; a yellow highlight indicates a digit which occurs in the secret square, but not in the same location as in the guess; and a grey highlight indicates a digit which does not occur in the secret square at all. The secret square may contain the same digit more than once.

In Squardle, only three attempts are allowed. Can you win the game which was started above?

Q1682 Use the Arithmetic–Logarithmic–Geometric Mean Inequality (see the article by Toyesh Prakash Sharma in this issue) to prove (without a calculator!) that

$$e^{2/\sqrt{5}} < \frac{\sqrt{5}+1}{\sqrt{5}-1} < e\,.$$

Q1683 For background on this problem, see the article by Timothy Hume in this issue. If A and B are points on a sphere, then we shall write $\widehat{AB}$ for the distance on the sphere between A and B. We denote the radius of the sphere by R.

(a) Use the coordinate formulae (2) and (3) in the article to prove the arc–length formula (8).

(b) Prove that if the great circle arcs AC and BC intersect at right angles, then

$$\cos\frac{\widehat{AB}}{R} = \left(\cos\frac{\widehat{AC}}{R}\right)\left(\cos\frac{\widehat{BC}}{R}\right).$$

(c) (For readers who have studied advanced calculus.) What do you get from the result of (b) if the arcs AB, BC and AC are very small compared with R?

Q1684

(a) Let $p(x) = 1 + 2x + 3x^2 + 4x^3$. Find a polynomial $q(x)$ with integer coefficients, not all zero, such that when $p(x)q(x)$ is expanded and terms collected, there will be no terms in x^k unless k is a square number. (Note that 0 is a square: so we want a product polynomial that looks like $a + bx + cx^4 + dx^9 + \cdots$.)

(b) Prove that if we replace $p(x)$ by any polynomial with integer coefficients, a polynomial $q(x)$ with this property can always be found.

Q1685

(a) Show how to arrange the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ around a circle in such a way that the sum of two neighbouring numbers is never a multiple of 3 or 5 or 7. In how many ways can this be done?

(b) Given any 9 consecutive integers, is the same task always possible?

Q1686 Simplify

$$\frac{\sqrt[3]{560 + 158\sqrt{2} + 324\sqrt{3} + 90\sqrt{6}}}{\sqrt[3]{560 - 158\sqrt{2} + 324\sqrt{3} - 90\sqrt{6}}}.$$

Q1687

(a) A line with gradient m intersects the ellipse $x^2 + 2y^2 = 3$ at the point $(1, 1)$ and another point. Find the other point.

(b) Find all triples of positive integers a, b, c having no common factor such that $a^2 + 2b^2 = 3c^2$.

Q1688 A one–person game is played as follows. Begin with a stack of n coins. Split them into two (non–empty) stacks with say a and b coins; this move gives a score of ab. Keep splitting the remaining stacks until all stacks consist of a single coin, and add all the scores. For example, starting with a stack of 30 coins, we might split it into stacks of 20 and 10, scoring 200; then into 20 and 7 and 3 scoring 21, total score so far 221; and so on until we have 30 stacks each containing one coin.

Prove that no matter how the coins are split, the final total score will always be the same.

Q1689 An ant walking across the floor noticed a grain of ant poison and a grain of sugar. Hating poison and loving sugar, the ant decided to walk in such a way that its distance from the poison increases at the same rate at which its distance to the sugar decreases. The ant was surprised to discover that no matter how fast it walked, it could not reach the sugar this way.

(a) Explain why the ant could not reach the sugar as long as it moved in the way described.

(b) Are there any exceptional cases when the ant could reach the sugar?

(c) Describe the path of the ant in case (a).

Q1690 In how many different ways can 10^{100} (a googol) be factorised as xyz, where x, y, z are positive integers,

(a) if the order of the factors does matter, for example, $2^{50} \times 5^{50} \times 10^{50}$ is regarded as different from $5^{50} \times 10^{50} \times 2^{50}$?

(b) if the order of the factors does not matter, for example, $2^{50} \times 5^{50} \times 10^{50}$ is the same as $5^{50} \times 10^{50} \times 2^{50}$?