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On a class of dense curves

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1 Introduction

A *Lissajous curve* is the graph of the system of parametric equations

$$x(t) = \cos(\alpha t)$$
$$y(t) = \cos(\beta t)\,.$$

One can prove that if α/β is rational, then the associated Lissajous curve is algebraic, i.e., it is the set of zeros of a polynomial.

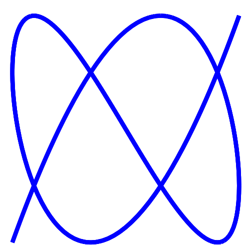

Figure 1: A Lissajous curve with $\alpha = 3$ and $\beta = 5$.

On the other hand, if α/β is irrational, then the associated curve is dense, meaning that it nearly fills the square $[-1, 1] \times [-1, 1]$. This is interesting as a curve is a one-dimensional figure while a square is two-dimensional. In mathematics, there are well-known examples of curves that completely fill a square, such as Peano curves. These are, however, not expressible by means of parametric equations and are obtained by considering a limit of functions, an infinite process.

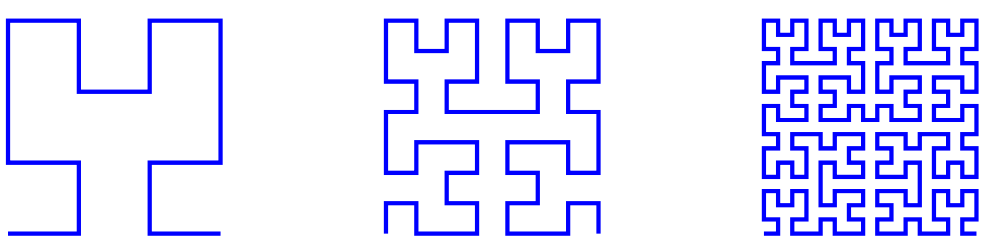

Figure 2: Some iterations in the construction of a Peano curve.

That is why in this paper, we focus on a class of curves that satisfy a weaker yet more straightforward condition, namely those that are dense in squares. In Proposition 8,

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the core result of this paper, we will show that any subset of the real line consisting of elements of the form $m + n\omega$, where ω is irrational and m and n are integers, is dense in the real line. This will allow us to show that if we replace the cosine function with another continuous, periodic function f, then the curve $\sigma(t) = f(\alpha t)\hat{e}_1 + f(\beta t)\hat{e}_2$ is dense in a suitable square, as we will demonstrate in Theorem 13. We can thus construct a wide variety of curves, all of which are dense in squares.

2 Background in Topology

Definition 1. *A subset U of $\mathbb{R}^2$ is* open *in $\mathbb{R}^2$ if, for every point $p \in U$, there exists $\epsilon > 0$ such that the ball*

$$B_\epsilon(p) = \{q \in \mathbb{R}^2 \; : \; d(p,q) < \epsilon\}$$

is contained in U.

The function d in the definition above is the usual Euclidean distance between points in the Cartesian plane. The following definition is a generalization of this that allows us to discuss open subsets in the Cartesian plane.

Definition 2. *Let X be a subset of $\mathbb{R}^2$. A subset $A \subseteq X$ is* open in X *if $A = U \cap X$ for some open subset $U \subseteq \mathbb{R}^2$.*

Example 1. Let $p \in X = \mathbb{R} \times \{0\}$ and consider an open ball $B = B_r(p)$. Then $B \cap X$ is a subset of the form $(x - \epsilon, x + \epsilon) \times \{0\}$ where x is the first coordinate of p and ϵ is a positive real number.

In what follows, we will identify $\mathbb{R}$ with $\mathbb{R} \times \{0\}$. Note that a subset U of $\mathbb{R}$ is open if and only if, for every $x \in U$, there exists $\epsilon \equiv \epsilon(x) > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$.

Definition 3. *Let $f : X \longrightarrow Y$ be a function between subsets X and Y of $\mathbb{R}^2$. We say that f is* continuous *if $f^{-1}(V)$ is open in X for every open subset V of Y.*

In other words, a function is continuous if arbitrarily small changes in its value can be assured by restricting it to small enough changes in its argument.

It is important to note that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if its inverse image $f^{-1}(a,b) = \{x \in X \; : \; f(x) \in (a,b)\}$ is open for every $a < b$. This follows from the fact that every open subset U of the real line can be written as

$$U = \bigcup_{i \in I} (a_i, b_i)$$

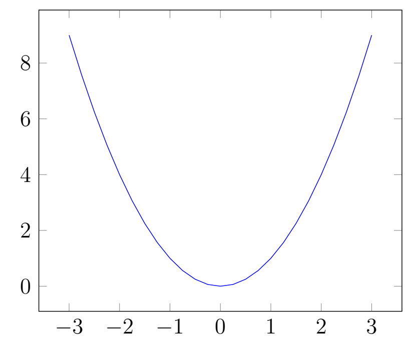
and

$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(a_i, b_i)\,.$$

We remind the reader that given a family of sets $\{A_i\}_{i \in I}$ indexed by an index set I, the symbol $\cup_{i \in I} A_i$ denotes the union of these sets:

$$\bigcup_{i \in I} A_i = \{x \; : \; x \in A_i \text{ for some } i \in I\}\,.$$

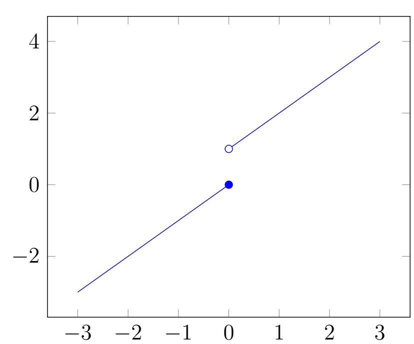
Example 2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. The function f is continuous. Indeed, we can prove this by considering two distinct cases. Let $(a, b) \subseteq \mathbb{R}$. If $a > 0$, then $f^{-1}(a, b) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$ which is open. If $a < 0$, then $f^{-1}(a, b) = (-\sqrt{b}, \sqrt{b})$ which is open.



Example 3. Let

$$f(x) = \begin{cases} x & \text{, if } x \leq 0 \\ x + 1 & \text{, if } x > 0\,. \end{cases}$$

The function $f(x)$ is not continuous. Indeed, the inverse image $f^{-1}(-1/2, 1/2)$ of the open interval $(-1/2, 1/2)$ is the interval $(-1/2, 0]$ which is not open. This follows from the fact that there exists no open interval centred at 0 and contained in $[0, 1/2)$.



Definition 4. *Let $X \subseteq \mathbb{R}^2$. A subset D of X is* dense *in X if, for every point $p \in X$ and every open subset A of X containing p, the intersection $A \cap D$ is nonempty.*

In other words, $D \subseteq X$ is dense in X if every point of X is arbitrarily close to D.

Example 4. Let $X = \mathbb{R}$ be the x-axis of the Cartesian plane $\mathbb{R}^2$ and $D = \mathbb{Q}$ be the set of rational numbers on the x-axis. It is well-known that D is a dense subset of X. Indeed, every real number can be approximated by a sequence of rational numbers.

Example 5. Let $X = [0, 1] \times [0, 1]$ and $D = \{(x, y) \in X \ : \ x \in \mathbb{Q}\}$. Then D is dense in X as described in the example above.

3 A Special Class of Real Subsets

The topology on the real line $\mathbb{R}$ results in subsets of $\mathbb{R}$ with a topological structure. The topological properties of subsets of the real line are closely related to their algebraic properties. This leads us to the following fundamental definition.

Definition 5. *A non-empty subset X of $\mathbb{R}$ is an additive subgroup if $x-y \in X$ for all $x, y \in X$.*

Note that each additive subgroup X of $\mathbb{R}$ contains 0, $-x$ and $x+y$ for all $x, y \in X$.

Example 6. Let $\mathbb{Z}$ be the subset of the real line consisting of all integer numbers. This is an additive subgroup. Indeed, if $a, b \in \mathbb{Z}$, then $a - b \in \mathbb{Z}$.

A feature of the set of integers is that its elements are "distanced" from one another. More formally, given any positive integer x, we can find a value $\epsilon > 0$ such that the open interval $(x-\epsilon, x+\epsilon)$ does not contain any other integer besides x. We can, for instance, fix $\epsilon = 1/2$. This property is what makes the subset of integers $\mathbb{Z}$ discrete in $\mathbb{R}$.

Definition 6. *Let X be a subset of $\mathbb{R}^2$. A subset Z of X is* discrete *in X if, for every $x \in Z$, there exists an open subset U_x of X such that $U_x \cap Z = \{x\}$.*

Note that, having identified $\mathbb{R}$ as $\mathbb{R} \times \{0\}$, a subset Z of $\mathbb{R}$ is discrete if, for every $x \in Z$, $\{x\} = (x-\epsilon, x+\epsilon) \cap Z$ for some $\epsilon > 0$.

Example 7. Let $\omega \in \mathbb{R}$ and $\Omega = \{m + n\omega \ : \ m, n \in \mathbb{Z}\}$ where ω is a rational number. It is clear that Ω is an additive subgroup of $\mathbb{R}$. We can write $\omega = a/b$ where a and b are integers with $b \neq 0$. Then each element x of Ω is of the form $x = m+na/b = (mb+na)/b$. Since $mb + na$ is an integer, x is an integer multiple of $1/b$. The elements of Ω are therefore at distance at least $1/b$ from each other, so, setting $\epsilon = 1/b$,

$$\{x\} = (x-\epsilon, x+\epsilon) \cap \Omega$$

for all $x \in \Omega$. This proves that Ω is discrete.

Theorem 7. *Each additive subgroup X of $\mathbb{R}$ is either dense or discrete.*

Proof. See [1]. □

The proof of the theorem above is not straightforward. That is why we present a weaker version of it below that is closely tied to the purposes of this paper.

Proposition 8. *For each $\omega \in \mathbb{R} \setminus \mathbb{Q}$, the following subset is dense in $\mathbb{R}$:*

$$\Omega = \{m + n\omega \ : \ m, n \in \mathbb{Z}\}.$$

To prove this proposition, we must first introduce a few concepts from topology.

Definition 9. *Let X be a subset of the real line $\mathbb{R}$. A point $p \in \mathbb{R}$ is a* limit point *of X if, for every $\epsilon > 0$, there exists a point $x \in X$, different from p, such that $x \in (p-\epsilon, p+\epsilon)$.*

That is, p is a limit point of X if we can find points in X arbitrarily close to p.

Example 8. Let $X = \{x_n = 1/n : n = 1, 2, 3, \ldots\} \subset \mathbb{R}$. Then 0 is a limit point of X. Indeed, if $\epsilon > 0$, then we can find a positive integer N such that $x_N = 1/N < \epsilon$. Thus,

$$|x_n - 0| = x_n < \epsilon$$

for every $n \geq N$.

Definition 10. *A subset X of $\mathbb{R}$ is* bounded *if $X \subseteq [-R, R]$ for some positive real number R. If $x \leq M$ for every $x \in X$, then M is an* upper bound *of X. Similarly, if $x \geq m$ for every $x \in X$, then m is a* lower bound *of X. If X has an upper bound, then the* supremum *$\sup X$ is the greatest number satisfying $\sup X \leq M$ for every upper bound M of X; otherwise, $\sup X = \infty$. Similarly, if X has a lower bound, then the infimum $\inf X$ is the smallest number satisfying $\inf X \geq m$ for every lower bound m of X; otherwise $\inf X = -\infty$.*

Example 9. If X is a bounded subset of $\mathbb{R}$, then either $S = \sup(x)$ and $s = \inf(X)$ are limits points of X, or they belong to X. It is enough to prove that this is true for $\sup(X)$ as the same arguments hold for $\inf(X)$. If $S \notin X$, then X does not contain any of its upper bounds. Let $\epsilon > 0$ and assume, by contradiction, that no point x of X belongs to $(S - \epsilon, S + \epsilon)$. This means, as S is an upper bound of X, that every element of $(S - \epsilon, S)$ is an upper bound of X, a contradiction since S is the smallest upper bound of X.

Lemma 11. *Let $X \subseteq \mathbb{R}$. If $\sup S$ and $\inf S$ belong to S for each non-empty subset S of X, then X is finite.*

Proof. Suppose that X is infinite and that $S = X$ contains $\inf S = \inf X$. Let $x_0 = \inf X$ and recursively define $x_{n+1} = \inf(X \setminus \{x_0, \ldots, x_n\})$. Now define the set

$$S' = \{x_0, x_1, x_2, \ldots\}$$

and note that S' is a subset of X. Since $x_0 < x_1 < x_2 < \cdots$, it follows that $\sup S'$ cannot belong to S'. We have proved that if X is infinite, then either S does not contain $\inf S$ or $S' = \{x_0, x_1, x_2, \ldots\}$ does not contain $\sup S'$. In either case, some subset of X does not contain both its infimum and its supremum. Therefore, if each subset S of X does contain both $\inf S$ and $\sup S$, then X cannot be infinite; X must be finite. □

We now present the Bolzano-Weierstrass Theorem. It is of fundamental importance to topology and real analysis.

Theorem 12 (Bolzano - Weierstrass)**.** *Let X be a bounded, infinite subset of the real line. Then X has a limit point.*

Proof. By Lemma 11, X contains a subset S that does not contain both $\inf S$ and $\sup S$. Suppose without loss of generality that S does not contain $\inf S$. Then, by Example 9, $\inf S$ is a limit point for S and thus for X. □

As we know, every real number x can be represented via its decimal expansion

$$x = r.\, r_1 r_2 r_3 \ldots$$

where $r \in \mathbb{Z}$ and $r_1, r_2, \ldots$ are integers between 0 and 9. This expansion is unique provided that the sequence r_n is not eventually constantly equal to 9. We can therefore treat every real number as its unique decimal expansion, which implies that two real numbers with two different decimal expansions are distinct.

The decimal expansion of any irrational number x is aperiodic. Indeed, if there existed positive integers $k < m$ such that

$$x = r.\, r_1 \ldots r_k (r_{k+1} \ldots r_m)(x_{k+1} \ldots r_m) \ldots ,$$

then

$$\begin{aligned} x10^k &= (r10^k + r_1 \ldots r_k) + 0.\, (r_{k+1} \ldots r_m)(r_{k+1} \ldots r_m) \ldots \\ x10^m &= (r10^m + r_1 \ldots r_m) + 0.\, (r_{k+1} \ldots r_m)(r_{k+1} \ldots r_m) \ldots , \end{aligned}$$

so

$$x = \frac{r(10^m - 10^k) + r_1 \ldots r_m - r_1 \ldots r_k}{10^m - 10^k} .$$

In particular, x is rational.

Proof of Proposition 8

Let ω be an irrational number and consider the set $\Omega = \{m + n\omega \ : \ m, n \in \mathbb{Z}\}$.

We first wish to show that $[0,1] \cap \Omega$ is infinite. Since ω is irrational, it can be represented as its decimal representation

$$\omega = r.\, r_1 r_2 \ldots$$

where r is an integer (which we can assume to be positive) and $r_1, r_2, \ldots$ are integers between 0 and 9. Note that for each positive integer k, the number

$$\omega_k = 0.\, r_k r_{k+1} \ldots = 10^k \omega - r - r_1 \ldots r_{k-1}$$

belongs to $[0,1] \cap \Omega$. Now, since the decimal representation of ω is not periodic, we can conclude that $\{\omega_k\}_{k \in \mathbb{N}} \subseteq [0,1] \cap \Omega$ is infinite. Indeed, the fact that the decimal representation of ω is not periodic implies that the sequence of truncated decimal representations ω_k contains an infinite number of distinct elements.

Now, by the Bolzano-Weierstrass Theorem, $\{\omega_1, \omega_2, \ldots\}$ has a limit point p; this is also a limit point for Ω. Then, given $\epsilon > 0$, there exist two distinct elements $x_1, x_2 \in \Omega$ such that $|x_1 - p| < \epsilon/2$ and $|x_2 - p| < \epsilon$. By the triangle inequality,

$$|x_1 - x_2| = |(x_1 - p) + (p - x_2)| \leq |x_1 - p| + |x_2 - p| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

Since $x = x_1 - x_2$ belongs to Ω, the argument above shows that for every $\epsilon > 0$, we can find an element $x \in \Omega$ with an absolute value less than ϵ.

Now, let y be any real number and let $\epsilon > 0$. Suppose that $y \geq 0$. By the above arguments, we can find $x \in \Omega$ such that $0 < |x| < \min(\epsilon, y)$. Since Ω is an additive subgroup of $\mathbb{R}$, it also contains $-x$, so we can assume that $0 < x < \min(\epsilon, y)$. Let n be the greatest positive integer such that

$$y = nx + \epsilon'$$

where $0 < \epsilon' < \epsilon$. As $x' = nx \in \Omega$ and $|y - x'| = |y - nx| < \epsilon' < \epsilon$, we have proved that there is an element x' of Ω that is at distance less than ϵ from y. A similar argument holds if $y < 0$. We have therefore proved that Ω contains points arbitrarily close to any real point. In other words, Ω is dense in $\mathbb{R}$. □

4 Density of Certain Curves

The following theorem is an interesting corollary of Proposition 8 and is the main result of this paper.

Theorem 13. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous, periodic function with period $P > 0$. If*

$$m = \min_{x \in \mathbb{R}} f(x) \text{ and } M = \max_{x \in \mathbb{R}} f(x)\,,$$

then the image of the curve

$$\sigma(t) = f(\alpha t)\hat{e}_1 + f(\beta t)\hat{e}_2$$

is dense in the square $Q = [m, M] \times [m, M]$ if α/β is irrational.
Here, $\hat{e}_1$ and $\hat{e}_2$ represent the vectors $(0, 1)$ and $(1, 0)$, respectively.

We prove this theorem later in this section.

Note that the function f in the theorem above is always bounded and must attain a maximum and a minimum on $\mathbb{R}$. This last fact follows from Theorem 14 below, first discovered and proved independently by Bolzana in the 1830s and by Weierstrass in 1960; see [2] for a proof. Note that this theorem applies to periodic continuous functions as well since their images are determined by an interval of the form $[a, b]$.

Theorem 14. *Each continuous function $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and attains a maximum value and a minimum value.*

Our strategy for proving Corollary 13 is straightforward. We will prove that the intersection of the image of f with every horizontal segment in the square Q is dense. This will be enough to check that the image of f is dense in Q since Q can be written as the union of horizontal segments $[m, M] \times \{y\}$ such that $m \leq y \leq M$.

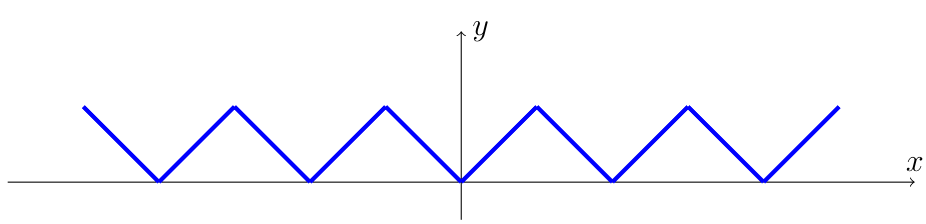


Figure 3: The graph of the periodic function $f(x) = \arccos(\cos(x))$.

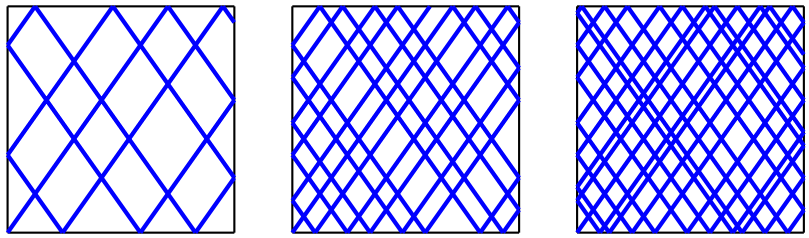


Figure 4: The curve $\sigma(t) = f(t)\hat{e}_1 + f(\sqrt{2}t)\hat{e}_2$ on the interval $[0, T]$ with $T = 5, 10, 15$.

Lemma 15. *A subset D of Q is dense if $D \cap L_y$ is dense in L_y for every $y \in [m, M]$ where $L_y = \{(x, y) \in Q \,:\, x \in [m, M]\}$.*

Proof. We need to show that for every $p \in Q$, there exists an open ball $B_\epsilon(p)$ such that $D \cap B_\epsilon(P) \neq \emptyset$. Let $p = (x, y) \in Q$, $\epsilon > 0$, and $B = B_\epsilon(p)$. By assumption, the subset $D \cap L_y$ is dense in L_y, so there must exist a value q belonging to $D \cap L_y$ such that $q \in B \cap L_y \subseteq B$. $\square$

Lemma 16. *Let X and Y be subsets of $\mathbb{R}^2$ and $f : X \longrightarrow Y$ be a continuous surjective function. If D is a dense subset of X, then*

$$f(D) = \{f(x) \,:\, x \in D\} \subseteq Y$$

is dense in Y.

Proof. Let $y \in Y$ and $U \subseteq Y$ be an open subset containing y. We need to show that there exists $p \in D$ such that $f(p) \in U$. Since f is surjective, there must exist $x \in X$ such that $y = f(x)$. Thus, the subset $f^{-1}(y) = \{x \in X : f(x) = y\}$ is nonempty. As $f^{-1}(y)$ is a subset of $f^{-1}(U)$, this latter set must be nonempty as well. Therefore, the subset $f^{-1}(U) \subseteq X$ is a nonempty, open subset of X. Since $D \subseteq X$ is dense in X, there exists $p \in D \cap f^{-1}(U) \subseteq D$. This shows that $f(p) \in f\big(D \cap f^{-1}(U)\big) \subseteq U$. $\square$

Proof of Theorem 13

We want to show that the subset

$$\sigma(\mathbb{R}) = \{\sigma(t) \,:\, t \in \mathbb{R}\} \subseteq Q$$

is dense in Q. Thanks to Lemma 15, we can check that $\sigma(\mathbb{R}) \cap L_y$ is dense in L_y for every $y \in [m, M]$. Since f is continuous, and the image of f is equal to $[m, M]$, there must exist a point $x \in Q$ such that $f(\beta x) = y$. This implies that $(f(\alpha x), y) \in L_y$. For every integer $k \in \mathbb{Z}$, let $x_k = x + kP/\beta$. Then

$$f(\beta x_k) = f(\beta x + \beta(k + (P/\beta)) = f(\beta x + kP) = f(\beta x) = y\,.$$

This shows that

$$\Omega' = \big\{\big(f(\alpha x_k + hP), y\big) \,:\, h, k \in \mathbb{Z}\big\}$$

is a subset of $\sigma(\mathbb{R}) \cap L_y$. Let $\omega = \alpha/\beta$. Then the set Ω' is the image of the subset

$$\Omega'' = \{(\alpha x + k\omega P + hP) \ : \ h, k \in \mathbb{Z}\} \subseteq \mathbb{R}$$

via the function $f : \mathbb{R} \longrightarrow L_y$ which takes a point $q \in \mathbb{R}$ to $(f(q), y) \in \Omega'$. Now, note that

$$\Omega'' = \alpha x + P \cdot \{k\omega + h \ : \ k, h \in \mathbb{Z}\}$$

which is the image of $\Omega = \{k\omega + h \ : \ k, h \in \mathbb{Z}\} \subseteq \mathbb{R}$ via the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(q) = \alpha x + Pq$ for all $q \in \mathbb{R}$. Thus, we can see that $\Omega' = (f \circ g)(\Omega)$. This also tells us that the function $f \circ g$ is continuous and surjective and that Ω is dense in $\mathbb{R}$ by Proposition 8. Thanks to Lemma 16, we can see that Ω' must be dense in Q. □

5 Acknowledgements

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References

[1] ProofWiki, Subgroup of Real Numbers Is Discrete or Dense, 26 Nov. 2020, https://proofwiki.org/wiki/Subgroup_of_Real_Numbers_is_Discrete_or_Dense, last accessed on 2022-12-19.

[2] W. Rudin, *Principles of Mathematical Analysis*, McGraw Hill, New York, 1976.