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Set theory as the foundation of mathematics with focus on the Axiom of Choice

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1 Introduction

The ultimate principles upon which mathematics rests are those to which mathematicians appeal without proof. The basic concepts of mathematics are those in terms of which all other mathematical concepts are ultimately defined [3]. Proof of these basic concepts provides a foundation for mathematics.

Therefore, for a theory to be a foundation of mathematics, it must provide a rigorous explanation of the nature of mathematical reality; mathematical reality implies a precise and formal definition, or representation of, basic mathematical concepts. A "foundation" of mathematics places emphasis on which formal systems allow us to formalize and prove the various theorems that will constitute mathematical truths [2]. In the last century, mathematicians have placed increased focus on axioms as the foundation of mathematics, creating these modern concepts and frameworks for mathematics to create a more rigorous understanding of mathematics. This means that one now may write down axioms and prove theorems from those axioms. Hence, if a theory is able to prove mathematical theorems using its axioms, it can be labelled as a foundation for mathematics.

In 1931, after Kurt Gődel's publication of his First Incompleteness Theorem proofs, a debate of which axioms to include for the foundations of mathematics began. Mathematicians had to choose systems of axioms that would create agreeable results and as many as possible. One such axiomatic system is know as the Zermelo-Frankel-Choice system, which includes the controversial Axiom of Choice. Many logicians were not satisfied by the inclusion of the Axiom of Choice, and were willing to disregard its positive results in order to avoid its controversial ones. Thus, the debate over the this axiom began.

In our papers, we outline an understanding of set theory and how it provides a foundation for mathematics as well as how the Axiom of Choice contributes to a stronger foundation. We describe the requirements of such a foundation and how the inclusion of axioms into the system is justified. Finally, we will explore and debate some of the controversial consequences of the Axiom of Choice in ZFC and develop reasoning for why the Axiom of Choice is still necessary.

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2 Background

Set theory is the study of sets. *Sets* are defined as collections of elements. Each *element* is some mathematical object, and each object may also be a set. Set theory began in earnest when Gregor Cantor showed that the number of points on a line could not be counted by the natural numbers [1] (imagine breaking the number line into a pile or group of points and how when one removes the points labelled with the natural numbers, it does not seem make a dent!). In other words, the set of all rational numbers $\mathbb{R}$ and the set of natural numbers $\mathbb{N}$ are not *bijective* (one cannot pair elements of each set together without having elements - in this case, numbers - left over) and must have different *cardinalities*, or comparatively different numbers of elements in each set. This led to the notion of different types of infinity. Infinite sets that can establish a bijection between their elements and the natural numbers are considered *countably infinite*. If there exists no such bijection, the set is considered *uncountably infinite*.

Kurt Gődel answered questions about the nature of mathematics using set theory in his incompleteness theorems. *Gődel's First Incompleteness Theorem* declares that a complete formal system of mathematics was impossible. *Complete systems* are ones in which all mathematical statements can be proved using the basic axioms of that system. So, in any formal system, there will be *undecidable propositions*: ones that can be neither proved true nor false within the system. *Gődel's Second Incompleteness Theorem* details that formal systems of mathematics cannot prove their own consistency. *Consistent systems* are ones that do not have contradictions - i.e., for any statement A, A and ¬A cannot both be true. Gődel's theorems disallow any sort of system of mathematics in which every mathematical statement is provable.

Zermelo, Skolem, and Frankel created certain axioms - the *Zermelo-Frankel Axioms (ZF)* - outlining the properties of sets in order to avoid the paradoxes that arise from self-reference (such as Russell's Paradox). These form the commonly-accepted foundations of modern set theory, as is discussed in detail in the following article.

In short, any sort of complete and consistent foundation of mathematics cannot exist. Instead, the theorems state that the best systems for a foundation, like ZFC, would be consistent but incomplete. This too requires some assumption, as by the second theorem, proving the consistency of ZFC is impossible using itself, and as the assumed foundation of mathematics, finding any sort of mathematical proof for its consistency would be impossible, as that proof could as easily be stated using ZFC.

3 Conclusion and Further Reading

In our papers, we outline an understanding of the foundation of mathematics, with particular emphasis on how set theory in general can provide a strong foundation for mathematics. For further reading on either subject, please see the article *The Axiom of Choice* in this issue of *Parabola*, on how this axiom interacts with the other axioms of the Zermelo-Fraenkel Axioms.

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