Parabola Volume 58, Issue 3 (2022)

On uses and applications of Muirhead's Inequality

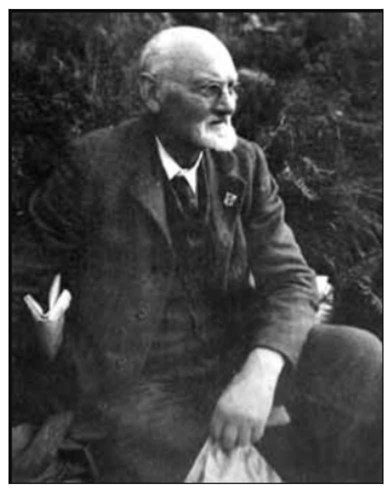
Soham Dutta[1]

1 Introduction

In this article, we will explore a less-known but very powerful class of inequalities known as *majorisation inequalities*. We will particularly focus on the inequality called *Muirhead's Inequality* and will present some of its applications to problems which have appeared in national and international maths olympiads and other maths competitions.

2 History

Muirhead's Inequality is named after and was discovered in 1901 by Robert Franklin Muirhead (1860-1941), pictured to the right[2], who was a mathematician born in Glasgow, Scotland. Muirhead graduated with a B.Sc. from the University of Glasgow in 1879 and with an M.A. in 1881, with the highest honours in mathematics and natural philosophy (physics). He was one of five students who graduated with an M.A. with Honours in Mathematics and Natural Philosophy in that year, two with First Class Honours and three with Second Class Honours.



Robert Franklin Muirhead

Throughout his career he published more than 90 papers, the most famous of them being that of Muirhead's Inequality under title "Inequalities relating to some algebraic means" [4].

[1]Soham Dutta is a 12-th grade high-school student studying in Delhi Public School Ruby Park, India.

[2]This picture was copied from the webpage
`https://mathshistory.st-andrews.ac.uk/Biographies/Muirhead/`.

3 Majorisation

Consider sequences $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ that are non-increasing:

$$x_1 \geq x_2 \geq \cdots \geq x_n$$
$$y_1 \geq y_2 \geq \cdots \geq y_n \, .$$

The sequence $\mathbf{x}$ *majorizes* $\mathbf{y}$, which we denote by

$$\mathbf{x} \succ \mathbf{y} \, ,$$

if and only if the following conditions hold:

$$\begin{aligned} & x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k \quad \text{for all } k = 1, 2, \ldots, n-1 \\ \text{and} \quad & x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n \, . \end{aligned}$$

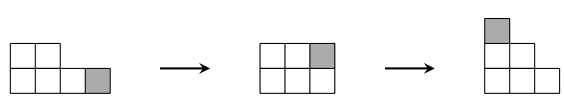
For example, $(3, 0) \succ (2, 1)$ since $3 \geq 2$ and $3 + 0 = 2 + 1$.
Also, $(1, 0, 0) \succ (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ since $1 \geq \frac{1}{3}$, $1 + 0 \geq \frac{1}{3} + \frac{1}{3}$ and $1 + 0 + 0 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$.

When the sequences $\mathbf{x}$ are integral, it can be useful to represent them visually as stacks of boxes; these are called *Young diagrams* or *Ferrers diagrams*. For instance, the Young diagrams for the sequences $(4, 2)$ and $(3, 3)$ are as follows:

These diagrams can be useful for visualising majorisation. In particular, a sequence $\mathbf{x}$ majorises another sequence $\mathbf{y}$ if it is possible to slide blocks from the diagram for $\mathbf{x}$ upwards to form the diagram for $\mathbf{y}$. For instance, the majorisations

$$(4, 2, 0) \succ (3, 3, 0) \succ (3, 2, 1)$$

can be visualised as follows:



We shall now state Muirhead's Inequality [4].

Theorem 1 (Muirhead's Inequality)**.**
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be non-increasing sequences with $\mathbf{x} \succ \mathbf{y}$. Then for all be non-negative real numbers $a_1, a_2, \ldots, a_n$,

$$\sum_{sym} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \geq \sum_{sym} a_1^{y_1} a_2^{y_2} \cdots a_n^{y_n} \, .$$

If $\mathbf{x} \neq \mathbf{y}$, then equality holds above if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

In the theorem above, the sums use *symmetric sum* notation, where the index runs over $n!$ permutations of the exponents $1, 2, \ldots, n$. For $n = 3$, examples of the use of this notation include

$$\sum_{\text{sym}} a^1b^3c^2 = a^1b^3c^2 + a^1b^2c^3 + a^2b^3c^1 + a^2b^1c^3 + a^3b^2c^1 + a^3b^1c^2$$

$$\sum_{\text{sym}} a^2b^2c^1 = a^2b^2c + b^2a^2c + b^2c^2a + c^2b^2a + c^2a^2b + a^2c^2b = 2(a^2b^2c + b^2c^2a + c^2a^2b)$$

$$\sum_{\text{sym}} a^3b^0c^0 = a^3b^0c^0 + a^3c^0b^0 + b^3c^0a^0 + b^3a^0c^0 + c^3a^0b^0 + c^3b^0a^0 = 2(a^3 + b^3 + c^3)\,.$$

In the last two examples, the sum runs over all $3! = 6$ permutations but since some terms of the sequences are equal, we get 3 pairs of equal terms. Similarly, we end up with just a single term in the following example:

$$\sum_{\text{sym}} a^1b^1c^1 = a^1b^1c^1 + a^1c^1b^1 + b^1a^1c^1 + b^1c^1a^1 + c^1a^1b^1 + c^1b^1a^1 = 6abc\,.$$

Muirhead's Inequality can be proved by induction on the number of terms of the sequences of powers. However, a proof will not be discussed here.

4 Some Useful Standard Inequalities

Here is a list of some common inequalities (without proofs) for later use in the article. These inequalities can for instance be found in [5].

QM-AM-GM-HM *Let $a_1, a_2, \ldots, a_n$ be n non-negative real numbers. Then*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}\,.$$

Equality occurs if $a_1 = a_2 = \cdots = a_n$.

Weighted AM-GM *Let $p_1, p_2, \ldots, p_n, a_1, a_2, \ldots, a_n \geq 0$ and $p_1 + p_2 + \cdots + p_n = p$. Then*

$$\frac{1}{p}\sum_{i=1}^{n} p_ia_i \geq \sqrt[p]{\prod_{i=1}^{n} a_i^{p_i}}\,.$$

Equality occurs if $a_1 = a_2 = \cdots = a_n$.

Cauchy-Schwarz *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \sum_{i=1}^{n}(a_ib_i)^2$$

Equality holds if $a_i = kb_i$ for some real k and all $i = 1, 2, \ldots, n$.

Titu's Lemma *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. Then*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \, .$$

Equality holds if $a_i = kb_i$ for some real k.

Rearrangement Inequality *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. Then*

$$\sum_{i=j} a_i b_j \geq \sum_{\substack{i \neq j \\ i+j \neq n+1}} a_i b_j \geq \sum_{i+j=n+1} a_i b_j \, .$$

Equality holds if $a_1 = a_2 = \cdots = a_n$.

Schur's Inequality *Let t be positive number and let x, y, z be non-negative numbers. Then*

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0 \, .$$

Equality holds if and only if $x = y = z$, or two of x, y, z are equal and the third equals 0.

5 Examples and Demonstrations

We now show how to use Muirhead's Inequality to solve assorted problems.

Example 1. *(Problem 1) For positive real numbers x, y, show that $x^5y^1 + y^5x^1 \geq x^4y^2 + y^4x^2$.*

We shall give three proofs to this problem: the first using the AM-GM Inequality, the second using elementary techniques and the third using Muirhead's Inequality.

Proof. (AM-GM) By the AM-GM Inequality,

$$\frac{x^4 + x^4 + x^4 + y^4}{4} \geq (x^{12}y^4)^{\frac{1}{4}} = x^3y$$
$$\frac{y^4 + y^4 + y^4 + x^4}{4} \geq (x^4y^{12})^{\frac{1}{4}} = y^3x \, ,$$

with equality exactly when $x = y$. Adding these inequality yields $x^4 + y^4 \geq x^3y + y^3x$. Since $xy > 0$, it follows that $x^5y^1 + y^5x^1 \geq x^4y^2 + y^4x^2$. □

Proof. (Elementary) Since x and y are both positive,

$$x^5y + y^5x - x^4y^2 - y^4x^2 = xy\,(x-y)^2(x^2 + xy + y^2) \geq 0 \, .$$

Hence, $x^5y + y^5x \geq x^4y^2 + y^4x^2$, and equality holds if and only if $x = y$. □

Proof. (Muirhead) Note that $(5, 1) \succ (4, 2)$, so apply Muirhead's Inequality on (x, y). Equality holds if and only if $x = y$. □

Example 2. *(Problem 2) Let a, b, c be positive real numbers. Show that*

$$\begin{aligned} & a^2+b^2+c^2 \geq ab+bc+ca \\ \textit{and} \qquad & a^3+b^3+c^3 \geq 3ab\,. \end{aligned}$$

We show three methods to solve this problem: the first using the AM-GM Inequality, the second using elementary methods and the third using Muirhead's Inequality.

Proof. (AM-GM) The AM-GM Inequality yields the following three inequalities:

$$\frac{a^2+b^2}{2} \geq ab\,, \qquad \frac{b^2+c^2}{2} \geq bc\,, \qquad \frac{c^2+a^2}{2} \geq ca\,.$$

Add these to obtain the problem's first inequality. Equality occurs when $a = b = c$.
To show the second inequality, apply the AM-GM Inequality to a^3, b^3, c^3. □

Proof. (Elementary) To show the first inequality, note that

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\big((a-b)^2+(b-c)^2+(c-a)^2\big) \geq 0\,, \tag{1}$$

with equality occurring if and only if $a = b = c$.
To show the second inequality, note that $a+b+c > 0$ and apply Inequality (1):

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq 0\,.$$

Equality holds if and only if $a = b = c$. □

Proof. (Muirhead) Observe that $(2,0,0) \succ (1,1,0)$ and $(3,0,0) \succ (1,1,1)$ and apply Muirhead's Inequality to (a, b, c). □

In the above example, we do not directly get inequalities $a^2+b^2+c^2 \geq ab+bc+ca$ and $a^3+b^3+c^3 \geq 3abc$ by applying Muirhead's Inequality. In particular, we get the inequalities $2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$ and $2(a^3+b^3+c^3) \geq 6abc$; first after dividing by 2 do we get the required results. This is an important fact that people overlook when applying Muirhead's Inequality. It is always a good practice to write out all terms without considering that they might get repeated.

Example 3. *(Problem 3) (Nesbitt) Show that, for positive real numbers a, b, c,*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}\,.$$

We show two methods to solve this problem: one using the AM-GM Inequality and the other using Muirhead's Inequality.

Proof. (AM-GM) Observe that the given inequality is equivalent to proving

$$\frac{2a+b+c}{b+c}+\frac{2b+a+c}{a+c}+\frac{2c+a+b}{a+b} \geq 6\,.$$

Now let $x = a + b$, $y = b + c$, $z = a + c$. Hence the above statement transforms to

$$\frac{x+z}{y} + \frac{y+z}{x} + \frac{x+y}{z} \geq 6$$

which is true since

$$\frac{x}{y} + \frac{z}{y} + \frac{y}{x} + \frac{z}{x} + \frac{x}{z} + \frac{y}{z} \geq 6$$

holds by the AM-GM Inequality, with equality if and only if $a = b = c$. □

Proof. (Muirhead) We can re-express the inequality as

$$2a^3 + 2b^3 + 2c^3 \geq ab^2 + a^2b + bc^2 + b^2c + c^2a + a^2c\,;$$

but this is true by Muirhead's Inequality since $(3, 0, 0) \succ (2, 1, 0)$. □

Example 4. *(Problem 4) (AM-GM) Show for positive real numbers* $a_1, a_2, \ldots, a_n$ *that*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}\,.$$

The classical proof of the AM-GM Inequality above is by Cauchy Induction. The proof of the inequality using Muirhead's Inequality is, however, a little tricky!

Proof. Observe that $(1, 0, \ldots, 0) \succ (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Thus we might apply Muirhead's Inequality on $(a_1, a_2, \ldots, a_n)$. But be careful!! The symmetric sum for the sequence $(1, 0, \ldots, 0)$ has $(n-1)!$ identical terms and the symmetric sum for $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ has $n!$ identical terms, all of which must be counted. By applying Muirhead's Inequality to $(a_1, a_2, \ldots, a_n)$, we see that

$$(n-1)!(a_1 + a_2 + a_3 + \cdots + a_n) \geq n!(a_1 a_2 \cdots a_n)^{\frac{1}{n}}\,.$$

Simplifying this inequality yields the required result. □

Let's now look at a few problems that have previously appeared in maths competitions.

Example 5. *(Problem 5) (Russia 1991) Let* a, b, c *be positive real numbers. Show that*

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}\,.$$

Proof. Clearing the denominators and multiplying both sides by

$$abc(a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc)$$

leads to the equivalent inequality

$$\sum_{\text{sym}} a^5 b^2 c^2 \leq \sum_{\text{sym}} a^6 b^3 c^0\,.$$

This is true by Muirhead's Inequality since $(6, 3, 0) \succ (5, 2, 2)$. □

The next problem is the famous inequality from the Iran 1996 competition.

Example 6. *(Problem 6) (Iran 1996) Prove, for positive real numbers a, b, c,*

$$(ab+bc+ca)\left(\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right) \geq \frac{9}{4}\,.$$

Proof. We can re-express this inequality as

$$4\sum_{\text{sym}} a^5b + \sum_{\text{sym}} a^4bc + 3\sum_{\text{sym}} a^2b^2c^2 \geq \sum_{\text{sym}} a^4b^2 + 2\sum_{\text{sym}} a^3b^2c + 3\sum_{\text{sym}} a^3b^3\,. \tag{2}$$

Since $(5,1,0) \succ (4,2,0)$ and $(5,1,0) \succ (3,3,0)$, Muirhead's Inequality implies that

$$3\sum_{\text{sym}} a^5b \geq 3\sum_{\text{sym}} a^3b^3 \qquad \text{and} \qquad \sum_{\text{sym}} a^5b \geq \sum_{\text{sym}} a^4b^2\,.$$

It now suffices to show that

$$\sum_{\text{sym}} a^4bc + 3\sum_{\text{sym}} a^2b^2c^2 - 2\sum_{\text{sym}} a^3b^2c \geq 0\,.$$

This is equivalent to showing

$$2abc(a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)) \geq 0\,.$$

This last inequality true by Schur's Inequality and hence we are done. □

We now look at Problem S600 [7] proposed by Adrian Andreescu.

Example 7. *(Problem 7) (Adrian Andreescu) Let a, b, c be positive real numbers. Show that*

$$\frac{8a}{3b^2+2bc+3c^2}+\frac{8b}{3c^2+2ac+3a^2}+\frac{8c}{3a^2+2ab+3b^2} \geq \frac{9}{a+b+c}\,.$$

Proof. We aim to show that

$$\frac{8a^2}{3ab^2+2abc+3ac^2}+\frac{8b^2}{3bc^2+2abc+3a^2b}+\frac{8c^2}{3ca^2+2abc+3cb^2} \geq \frac{9}{a+b+c}\,.$$

By Titu's Lemma, we have

$$\frac{8a^2}{3ab^2+2abc+3ac^2}+\frac{8b^2}{3bc^2+2abc+3a^2b}+\frac{8c^2}{3ca^2+2abc+3cb^2} \geq \frac{8(a+b+c)^2}{6abc+3\sum_{\text{sym}} ab^2}\,.$$

It therefore suffices to show that

$$\frac{8(a+b+c)^2}{3ab^2+3a^2b+6abc+3ac^2+3bc^2+3b^2c+3a^2c} \geq \frac{9}{a+b+c}\,.$$

By cross-multiplying and cancelling out like terms, we have to show that

$$8a^3+8b^3+8c^3 \geq 3ab^2+3a^2b+6abc+3ac^2+3bc^2+3b^2c+3a^2c\,.$$

Since $(3,0,0) \succ (2,1,0)$ and $(3,0,0) \succ (1,1,1)$, Muirhead's Inequality applied to (a,b,c) gives

$$\begin{aligned} 2a^3+2b^3+2c^3 &\geq 6\,abc \\ \text{and} \qquad 3(2a^3+2b^3+2c^3) &\geq 3(ab^2+a^2b+ac^2+bc^2+b^2c+a^2c)\,. \end{aligned}$$

Adding the above inequalities yields the result. □

The following problem is by Nguyen Viet Hung and was published in Mathematical Reflections.

Example 8. *(Problem 8) Prove that, for positive real numbers a, b, c,*

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}\,.$$

Proof. By clearing denominators, we see that it suffices to show

$$(a^4+b^4+c^4)(a^2+b^2+c^2) \geq 3abc(a^3+b^3+c^3)$$

or, equaivalently,

$$\sum_{\text{sym}} a^6b^0c^0 + 2\sum_{\text{sym}} a^4b^2c^0 \geq 3\sum_{\text{sym}} a^4bc\,.$$

Since $(6,0,0) \succ (4,1,1)$ and $(4,2,0) \succ (4,1,1)$, Muirhead's Inequality applied to (a,b,c) gives

$$\sum_{\text{sym}} a^6b^0c^0 \geq \sum_{\text{sym}} a^4bc \qquad \text{and} \qquad 2\sum_{\text{sym}} a^4b^2c^0 \geq 2\sum_{\text{sym}} a^4bc\,.$$

Adding these two gives our required inequality, with equality when $a = b = c$. □

6 Some Words on the Usage of Muirhead's Inequality

Hopefully, it is by now clear to the reader that Muirhead's Inequality is nothing but a stronger version of the AM-GM Inequality. In fact, one can even boldly say that all problems that can be solved by Muirhead's Inequality can always be solved using just the AM-GM Inequality and the Weighted AM-GM Inequality.

The reader must also note that they should not apply Muirhead's Inequality to each and every problem they face at the very beginning. It is generally not advisable to use Muirhead's Inequality in mathematical contests or competitions. However, if they are not at all able to find a solution using standard inequalities like the AM-GM Inequality or the Cauchy-Schwarz Inequality, or if they under time pressure, then it can sometimes be useful to use Muirhead's Inequality.

A common usage area of Muirhead's Inequality is problems that involve fractional terms. In order to apply Muirhead's Inequality directly to such cases, one must multiply the terms together and clear the denominators. This process is invariably prone to calculation errors, and the reader should be very careful in their calculations.

7 A Common Pitfall

Let's say you have to prove

$$a^7 + b^7 + c^7 \geq a^4b^3 + b^4c^3 + c^4a^3 .$$

One may exclaim that since $(7, 0, 0) \succ (4, 3, 0)$, then by applying Muirhead to (a, b, c), we are done. This is a wrong proof. Why? Because Muirhead's Inequality works only for symmetric sums, and

$$\sum_{\text{sym}} a^4b^3c^0 = a^4b^3c^0 + a^4c^3b^0 + b^4a^3c^0 + b^4c^3a^0 + c^4a^3b^0 + c^4b^3a^0 .$$

Clearly, this is not the left-hand side of the inequality above. A correct solution to this problem is to observe that the AM-GM Inequality implies that

$$\begin{aligned}
a^7 + a^7 + a^7 + a^7 + b^7 + b^7 + b^7 &\geq 7a^4b^3 \\
b^7 + b^7 + b^7 + b^7 + c^7 + c^7 + c^7 &\geq 7b^4a^4 \\
c^7 + c^7 + c^7 + c^7 + a^7 + a^7 + a^7 &\geq 7c^4a^3 .
\end{aligned}$$

Adding these inequalities yields the required inequality above.

Another example of such a pitfall is to prove, for positive real numbers a, b, c, that

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a .$$

A wrong solution is to note that $(3, 0, 0) \succ (2, 1, 0)$ and apply Muirhead's Inequality to (a, b, c). A correct solution is to observe that the AM-GM Inequality implies

$$\begin{aligned}
a^3 + a^3 + b^3 &\geq 3a^2b \\
b^3 + b^3 + c^3 &\geq 3b^2c \\
c^3 + c^3 + a^3 &\geq 3c^2a .
\end{aligned}$$

Adding these yields the required result. Equality holds if and only if $a = b = c$.

One should always be careful in such kind of questions and pitfalls and carefully check all the terms on both the right-hand side and the left-hand side of the inequality, to see whether they really constitute a symmetric sum or not.

8 Problems for Practice

Readers are invited to try the following problems. All problems admit non-Muirhead solutions but they can be a little tricky to find in certain cases. In most of the problems, it is better to not apply Muirhead's Inequality directly but to simplify the problem using the methods given in Section 4 and then finish the problem using Muirhead's Inequality. The problems are not arranged according to difficulty.

Problem 1. (IMO 2005) *Let real numbers $x, y, z > 0$ satisfy $xyz \geq 1$. Prove that*

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + x^2 + z^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0\,.$$

Problem 2. (IMO 1984) *Let a, b, c be positive real numbers with $a + b + c = 1$. Show that*

$$0 \leq ab + bc + ca - 2abc \leq \frac{7}{27}\,.$$

Problem 3. (IMO 1995) *Let a, b, c be positive real numbers with $abc = 1$. Show that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}\,.$$

Problem 4. (IMO Shortlist 1998) *For real numbers $x, y, z > 0$ and $xyz = 1$ prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}\,.$$

Problem 5. *Prove that, for real numbers $a, b, c > 0$,*

$$(a + b - c)(b + c - a)(c + a - b) \leq abc\,.$$

Problem 6. (IberoAmerican Shortlist 2003) *Prove that, for real numbers $a, b, c > 0$,*

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}\,.$$

Problem 7. (IMO 1964) (Weizenböck) *Let a, b, c be the side lengths of a triangle and let Δ denote its area. Show that*

$$4\sqrt{3}\Delta \leq a^2 + b^2 + c^2\,.$$

Problem 8. (Nguyen Viet Hung) [7] *Let a, b, c be the sides of a triangle. Show that*

$$\frac{a}{b + c - a} + \frac{b}{c + a - b} + \frac{c}{a + b - c} \geq \frac{3(a^2 + b^2 + c^2)}{ab + bc + ca}\,.$$

Problem 9. *For non-negative real numbers a, b, c show that*

$$a^3 + b^3 + c^3 + abc \geq \frac{1}{7}(a + b + c)^3\,.$$

Problem 10. (Mihaly Bencze, Neculai Stanciu) [7]
Let ABC be a triangle with side-lengths a, b, c. Prove that

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \geq 4 - \frac{2r}{R}$$

where r and R are the in-radius and circum-radius of the triangle, respectively.

Problem 11. (Marin Chirciu) *For a triangle ABC prove that*

$$\frac{h_b}{h_a^2} + \frac{h_c}{h_b^2} + \frac{h_a}{h_c^2} \geq 3r\left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}\right).$$

where h_a, h_b, h_c denotes the respective altitudes from vertices A, B, C onto the opposite sides and r is the in-radius.

Problem 12. (AoPS) *Let x, y, z be positive real numbers such that $(x+y)(y+z)(z+x) = 8$. Prove that*

$$x^3y^3 + y^3z^3 + z^3x^3 + x^2y^2z^2 \geq 4xyz.$$

Problem 13. (IMO Longlist 1967) *Prove for positive numbers a, b, c that*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^8 + b^8 + c^8}{a^3b^3c^3}.$$

Problem 14. (Mock USAJMO 2015 Shortlist) *Let x, y, z be positive real numbers. Prove that*

$$(x^2 + y^2 + z^2)^2 \geq 3xyz(x + y + z).$$

Problem 15. (AoPS) *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^2 + bc}{b^2 + bc + c^2} + \frac{b^2 + ca}{c^2 + ca + a^2} + \frac{c^2 + ab}{a^2 + ab + b^2} \geq 2.$$

Problem 16. (Mathcenter 2012 Thailand) *Let $a, b, c \geq 0$ and $abc = 1$. Prove that*

$$\frac{a}{b^2(c+a)(a+b)} + \frac{b}{c^2(a+b)(b+c)} + \frac{c}{a^2(c+a)(a+b)} \geq \frac{3}{4}.$$

Problem 17. (Pham Kim Hung) *Let a, b, c be non-negative real numbers with $a + b + c = 3$. Show that*

$$\frac{a^2b}{4 - bc} + \frac{b^2c}{4 - ca} + \frac{c^2a}{4 - ab} \leq 1.$$

Problem 18. *Let x, y, z be positive real numbers such that $xyz = 1$. Show that*

$$x + y + z \geq \frac{3}{x+2} + \frac{3}{y+2} + \frac{3}{z+2}.$$

Problem 19. *Let x, y, z be positive real numbers such that $x + y + z = 2$. Prove that*

$$\frac{x^2\sqrt{y}}{\sqrt{x+z}} + \frac{y^2\sqrt{z}}{\sqrt{y+x}} + \frac{z^2\sqrt{x}}{\sqrt{z+y}} \leq \sqrt{x^3 + y^3 + z^3}.$$

Problem 20. *Let a, b, c be positive real numbers satisfying $a + b + c = a^2 + b^2 + c^2$. Show that*

$$ab + bc + ca \leq abc + 2.$$

9 Acknowledgement

I am heavily indebted and grateful to the entire Parabola Magazine team and in particular to Thomas Britz, David Angell and Sin Keong Tong who found time to read my article and suggest many valuable changes.

References

[1] The Art Of Problem Solving Community,
https://artofproblemsolving.com/community/,
last accessed on 2022-12-23.

[2] E. Chan, *A Brief Introduction to Olympiad Inequalities*,
https://web.evanchen.cc/handouts/Ineq/en.pdf,
last accessed on 2022-12-23.

[3] T. Mildorf, *Olympiad Inequalities*,
https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf,
last accessed on 2022-12-23.

[4] R. Muirhead, Inequalities relating to some algebraic means,
Proceedings of the Edinburgh Mathematical Society **19** (1900), 36–45.

[5] S. Riasat, *Basics of Olympiad Inequalities*,
https://web.williams.edu/Mathematics/sjmiller/public_html/161/articles/Riasat_BasicsOlympiadInequalities.pdf,
last accessed on 2022-12-23.

[6] B.J. Venkatachala, *Inequalities: An Approach Through Problems*,
Hindustan Book Agency, 2015.

[7] Senior Problems, *Mathematical Reflections* **4** (2022),
https://www.awesomemath.org/wp-pdf-files/math-reflections/mr-2022-04/mr_4_2022_problems.pdf,
last accessed on 2022-12-23.