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# Finding the exact value of $\sin 1^\circ$

**Kyumin Nam**[1]

## 1 Introduction

The exact values of sine such as $\sin 30^\circ = 1/2$ and $\sin 45^\circ = 1/\sqrt{2}$ are well known, but the exact value for sine for other angels such as $\sin 1^\circ$ and $\sin 7^\circ$ are not widely known. In this note, we find the exact values for $\sin 15^\circ$, $\cos 15^\circ$, $\sin 18^\circ$, $\cos 18^\circ$, $\sin 3^\circ$ and $\sin 1^\circ$ using angle sum and difference identities and triple-angle formulas.

We first present and prove these identities and formulas using Euler's Formula [1]:

$$e^{i\theta} = \cos\theta + i\,\sin\theta\,.$$

**Theorem 1** (Angle sum and difference identities). *For all* $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\,,\\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\,.\end{aligned}$$

*Proof.* By Euler's Formula,

$$\begin{aligned}\cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta) &= e^{i(\alpha\pm\beta)} = e^{i\alpha}e^{\pm i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta \pm i\sin\beta)\\ &= (\cos\alpha\cos\beta \mp \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta \pm \cos\alpha\sin\beta).\end{aligned}$$

Comparing the real part and the imaginary part completes the proof. □

The following double-angle formulas follow as corollary from Theorem 1.

**Theorem 2** (Double-angle formulas). *For each* $\alpha \in \mathbb{R}$,

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ \cos 2\alpha &= 1 - 2\sin^2\alpha\,.\end{aligned}$$

**Theorem 3** (Triple-angle formulas). *For all* $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\sin 3\alpha &= -4\sin^3\alpha\ + 3\sin\alpha\\ \cos 3\alpha &= \quad 4\cos^3\alpha - 3\cos\alpha\,.\end{aligned}$$

[1]Kyumin Nam is a student at Incheon Shinjeong Middle School.

*Proof.* By Euler's Formula,

$$\begin{aligned}\cos 3\alpha + i\sin 3\alpha = e^{3i\alpha} = (e^{i\alpha})^3 &= (\cos\alpha + i\sin\alpha)^3 \\ &= (\cos^3\alpha - 3\cos\alpha\sin^2\alpha) + i(3\sin\alpha\cos^2\alpha - \sin^3\alpha) \\ &= \big(\cos^3\alpha - 3\cos\alpha(1-\cos^2\alpha)\big) + i\big(3\sin\alpha(1-\sin^2\alpha) - \sin^3\alpha\big) \\ &= (4\cos^3\alpha - 3\cos\alpha) + i(3\sin\alpha - 4\sin^3\alpha)\,.\end{aligned}$$

Comparing the real part and the imaginary part completes the proof. □

We will also state, without proof, the following theorem from [2].

**Theorem 4.** *The roots of each cubic polynomial* $x^3 + a_2x^2 + a_1x + a_0$ *are*

$$\begin{aligned}x_1 &= -\frac{1}{3}a_2 + S + T \\ x_2 &= -\frac{1}{3}a_2 - \frac{1}{2}(S+T) + \frac{\sqrt{3}}{2}\,i\,(S-T) \\ x_3 &= -\frac{1}{3}a_2 - \frac{1}{2}(S+T) - \frac{\sqrt{3}}{2}\,i\,(S-T)\,,\end{aligned}$$

*where*

$$\begin{aligned}S &= \sqrt[3]{R + \sqrt{Q^3 + R^2}} \\ T &= \sqrt[3]{R - \sqrt{Q^3 + R^2}}\end{aligned}$$

*and*

$$Q = \frac{3a_1 - a_2^2}{9} \qquad \textit{and} \qquad R = \frac{9a_2a_1 - 27a_0 - 2a_2^3}{54}\,.$$

## 2 Sine and cosine derivations

**Lemma 5.**

$$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4} \qquad \textit{and} \qquad \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}\,.$$

*Proof.* According to angle difference identity, $\sin 15^\circ$ equals

$$\sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}\,.$$

Similarly, $\cos 15^\circ$ equals

$$\cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\,.$$

This completes the proof. □

**Lemma 6.**

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} \qquad \textit{and} \qquad \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

*Proof.* Let $\theta = 18^\circ$. Then $3\theta = 90^\circ - 2\theta$, so

$$\sin 3\theta = \sin(90^\circ - 2\theta) = \cos(2\theta) = 1 - 2\sin^2\theta.$$

Also,

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

by the triple-angle formula, so

$$1 - 2\sin^2\theta = \sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Letting $x = \sin\theta$, we have

$$x^3 - \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{4} = 0.$$

By Theorem 4, the three solutions to this equation are

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -\frac{\sqrt{5}+1}{4} \\ x_3 &= \phantom{-}\frac{\sqrt{5}+1}{4}. \end{aligned}$$

Since $\sin 0^\circ = 0 < \sin 18^\circ < \sin 30^\circ < \frac{1}{2}$,

$$\sin\theta = \frac{\sqrt{5}-1}{4}.$$

Since $\cos^2\theta = 1 - \sin^2\theta$ by Theorem 2,

$$\cos\theta = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

This completes the proof. □

**Lemma 7.**

$$\sin 3^\circ = \frac{\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}-2\sqrt{15+3\sqrt{5}}+2\sqrt{5+\sqrt{5}}}{16}.$$

*Proof.* According to angle difference identity, we get

$$\begin{aligned} \sin 3^\circ &= \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ \\ &= \frac{\sqrt{5}-1}{4}\frac{\sqrt{6}+\sqrt{2}}{4} - \frac{\sqrt{10+2\sqrt{5}}}{4}\frac{\sqrt{6}-\sqrt{2}}{4}. \end{aligned}$$

Expanding this expression completes the proof. □

**Proposition 8.**

$$\sin 1^\circ = \frac{-1-\sqrt{3}i}{4}\sqrt[3]{-\alpha+\sqrt{\alpha^2-1}} + \frac{-1+\sqrt{3}i}{4}\sqrt[3]{-\alpha-\sqrt{\alpha^2-1}}$$

*where*

$$\alpha = \sin 3^\circ = \frac{\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}-2\sqrt{15+3\sqrt{5}}+2\sqrt{5+\sqrt{5}}}{16}\,.$$

*Proof.* According to the triple-angle formula,

$$\sin 3^\circ = 3\sin 1^\circ - 4\sin^3 1^\circ\,.$$

Let $x = \sin 1^\circ$ and $\alpha = \sin 3^\circ$. Then $3x - 4x^3 = \alpha$, so

$$x^3 - \frac{3}{4}x + \frac{\alpha}{4} = 0\,.$$

By Theorem 4,

$$\begin{aligned}
x_1 &= \phantom{\omega^2}\frac{\sqrt[3]{-\alpha+\sqrt{\alpha^2-1}}}{2} + \phantom{\omega^2}\frac{\sqrt[3]{-\alpha-\sqrt{\alpha^2-1}}}{2} \approx \phantom{-}0.857\\
x_2 &= \omega\,\frac{\sqrt[3]{-\alpha+\sqrt{\alpha^2-1}}}{2} + \omega^2\frac{\sqrt[3]{-\alpha-\sqrt{\alpha^2-1}}}{2} \approx -0.875\\
x_3 &= \omega^2\frac{\sqrt[3]{-\alpha+\sqrt{\alpha^2-1}}}{2} + \omega\,\frac{\sqrt[3]{-\alpha-\sqrt{\alpha^2-1}}}{2} \approx \phantom{-}0.017
\end{aligned}$$

where $\omega = e^{2\pi i/3} = \dfrac{-1+\sqrt{3}i}{2}$. Since $\sin 0^\circ = 0 < \sin 1^\circ < \sin 30^\circ = \dfrac{1}{2}$,

$$\sin 1^\circ = \omega^2\,\frac{\sqrt[3]{-\alpha+\sqrt{\alpha^2-1}}}{2} + \omega\,\frac{\sqrt[3]{-\alpha-\sqrt{\alpha^2-1}}}{2}\,.$$

This completes the proof. □


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# References


[1] E.W. Weisstein, Multiple-angle formulas, *MathWorld–A Wolfram Web Resource*, https://mathworld.wolfram.com/Multiple-AngleFormulas.html, last accessed on 22 December 2022.

[2] E.W. Weisstein, Cubic formula, *MathWorld–A Wolfram Web Resource*, https://mathworld.wolfram.com/CubicFormula.html, last accessed on 22 December 2022.