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# An infinitely bounded region with finite area

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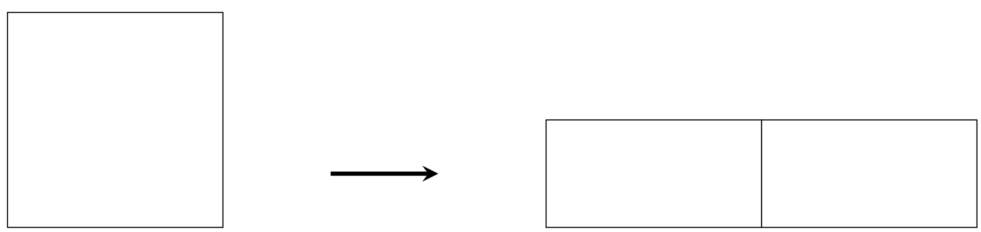
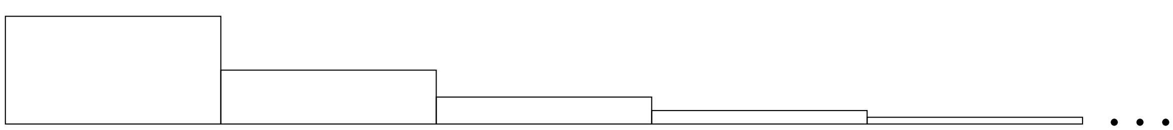
## 1 Introduction

The concept of improper integrals, or integrals with either infinite limits of integration or an integrand that approaches infinity, can be initially paradoxical. It's difficult to comprehend how the area underneath some curves that continue towards infinity could have anywhere near a finite value.

A nice example to try and comprehend this involves cutting a square in half. If we take a square with some finite area, say 16 square units for our purposes, then we can cut the square in half, putting the two halves next to one another:



Despite the length of this new shape being longer than our original square, the area of the shape remains the same. If we continue this process, cutting the last rectangle and placing the two halves side by side infinitely many times, then, despite our length tending towards infinity, our area remains 16 square units. Though the length is increasing quickly towards infinity, the height is decreasing at an equally fast pace, allowing us to retain a finite area.



Considering the concept of improper integrals, it's reasonable to wonder why some result in finite areas while others result in infinite ones. Furthermore, if an improper integral is infinite in one direction, say the $x$ direction, then what's preventing it from being infinite in both directions, yet retaining a finite area?

In order to approach this problem, a good starting place is looking at functions shaped similar to that of our rectangle example above. If one variable is approaching infinity, or forming an asymptote, then the other needs to approach zero just as fast or

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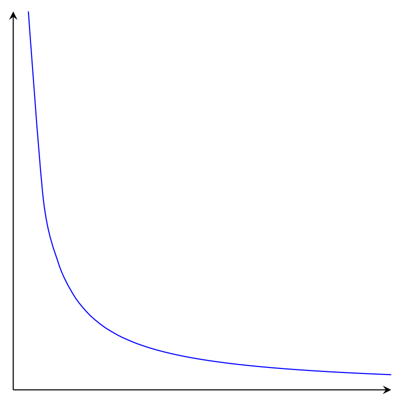
the area will be infinite. Because we're looking to approach infinity in both the $x$ and $y$ directions, we need a function with asymptotes on both axis.

One function that comes to mind is $y = \frac{1}{x}$, as shown below. While it has the shape we want, because its antiderivative results in $y = \ln x$, whose $y$-value approaches infinity as $x$ approaches infinity, the area is not finite.



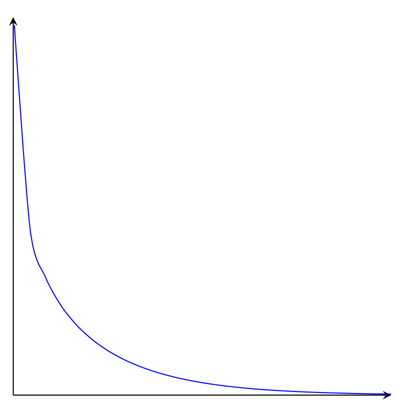
Exploring other similarly shaped functions, the term involution becomes significant. An involution is essentially a function that is an inverse of itself, i.e. where in all circumstances, $f(x) = f^{-1}(x)$; see [1]. Functions like $f(x) = \frac{1}{x}$, and others with a similar shape with asymptotes on both axis, are considered to be involutions.

At this point, having found a category representing the type of functions we're looking for, the goal now becomes finding a function whose antiderivative, or area, as $x$ approaches infinity, does not also approach infinity.

These criteria led me to the function

$$f(x) = \ln\left(\frac{e^x + 1}{e^x - 1}\right) .$$

Despite having a strikingly similar shape to $f(x) = \frac{1}{x}$, its area from $x = 0$ to $x = \infty$ is surprisingly finite.



## 2 Using a u-substitution

Our integration is as follows:

$$\lim_{b\to\infty}\int_0^b \ln\left(\frac{e^x+1}{e^x-1}\right)dx\,.$$

In order to make this integration easier, we can modify the argument of the natural log:

$$\frac{e^x+1}{e^x-1}=\frac{\frac{e^x}{e^x}+\frac{1}{e^x}}{\frac{e^x}{e^x}-\frac{1}{e^x}}=\frac{1+e^{-x}}{1-e^{-x}}\,.$$

Therefore,

$$\lim_{b\to\infty}\int_0^b \ln\left(\frac{e^x+1}{e^x-1}\right)dx=\lim_{b\to\infty}\int_0^b \ln\left(\frac{1+e^{-x}}{1-e^{-x}}\right)dx\,.$$

Using a u-substitution, we can simplify the integral to apply a new integration technique. Letting $u=e^{-x}$, we can find that: $\frac{du}{dx}=-e^{-x}$ and thus

$$dx=\frac{-1}{e^{-x}}du=\frac{-1}{u}du\,.$$

Substituting these values into the original integral, we can see that

$$\lim_{b\to\infty}\int_0^b \ln\left(\frac{e^x+1}{e^x-1}\right)dx=\lim_{b\to\infty}\int_{x=0}^{x=b}\left(\frac{-1}{u}\right)\ln\left(\frac{1+u}{1-u}\right)du.$$

Before integrating, we can change the bounds of this existing integral to be in terms of $u$ rather than in terms of $x$. Letting $x=0$, we can see that

$$u=e^{-x}=e^0=1\,.$$

Letting $x=b$, we can see that:

$$u=e^{-x}=e^{-b}=\frac{1}{e^b}\,.$$

Dealing with the limit out front makes our integration simpler later:

$$\lim_{b\to\infty}\frac{1}{e^b}=0\,.$$

Therefore, the integral becomes

$$-\int_1^0\left(\frac{1}{u}\right)\ln\left(\frac{1+u}{1-u}\right)du=\int_0^1\left(\frac{1}{u}\right)\ln\left(\frac{1+u}{1-u}\right)du\,.$$

To simplify the integral even further, we can rewrite our natural logarithm, allowing us to separate the original integral into two separate integrals:

$$\begin{aligned}\int_0^1\left(\frac{1}{u}\right)\ln\left(\frac{1+u}{1-u}\right)\,du&=\int_0^1\left(\frac{1}{u}\right)\left(\ln(1+u)-\ln(1-u)\right)\,du\\&=\int_0^1\left(\frac{1}{u}\right)\ln(1+u)\,du-\int_0^1\left(\frac{1}{u}\right)\ln(1-u)\,du\,.\end{aligned}$$

## 3 The polylogarithm as an integration technique

Just looking at these integrals, it's difficult to ascertain the best integration method going forward. A representation of the natural logarithm function that's helpful to us in this circumstance is the polylogarithm. The polylogarithm, denoted as $\mathrm{Li}_n(x)$, is the function

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}\,.$$

The polylogarithm is particularly valuable to us, as $\mathrm{Li}_1(x) = -\ln(1-x)$; see [2]. Making this substitution, we can conclude that:

$$\mathrm{Li}_1(x) = -\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}\,.$$

Substituting $u$ for $x$, we can therefore substitute this sum in for our first natural log in our integral:

$$\int_0^1 \left(\frac{1}{u}\right) \sum_{k=1}^{\infty} \frac{u^k}{k}\,du\,.$$

By the Second Constant Rule of Summation Algebra, constants within a summation can be pulled out front; see[3]. We can therefore do the opposite of this, pulling the $\frac{1}{u}$ inside the summation and simplifying:

$$\int_0^1 \sum_{k=1}^{\infty} \frac{u^k}{uk}\,du = \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k}\,du\,.$$

Now, taking a look at the other natural log within our integral, $\ln(1+u)$, we can also apply a polylogarithm. Substituting $-u$ into our polylogarithm when $n = 1$ gives

$$Li_1(-u) = -\ln(1-(-u)) = -\ln(1+u)\,.$$

Putting this in summation form, we find that

$$-Li_1(-u) = \ln(1+u) = -\sum_{k=1}^{\infty} \frac{(-u)^k}{k}\,.$$

Again, going back to the Second Constant Rule, we can bring our $\frac{1}{u}$ inside our integral:

$$-\int_0^1 \left(\frac{1}{u}\right) \sum_{k=1}^{\infty} \frac{(-u)^k}{k}\,du = \int_1^0 \sum_{k=1}^{\infty} \frac{(-u)^k}{uk}\,du = \int_1^0 \sum_{k=1}^{\infty} \frac{(-1)^k u^k}{uk}\,du = \int_1^0 \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k}\,du\,.$$

Now that we've made both substitutions, we can bring our two integrals back together again:

$$\int_0^1 \left(\frac{1}{u}\right) \ln(1+u)\,du - \int_0^1 \left(\frac{1}{u}\right) \ln(1-u)\,du = \int_1^0 \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k}\,du + \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k}\,du\,.$$

## 4 Integrating within a summation

Looking at the first of these integrals, we can take the integral of our summation:

$$\int_1^0 \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k}\, du = \sum_{k=1}^{\infty} \int_1^0 \left(-1 + \frac{u}{2} - \frac{u^2}{3} + \frac{u^3}{4} - \cdots\right) du$$
$$= \sum_{k=1}^{\infty} \left[-u + \frac{u^2}{4} - \frac{u^3}{9} + \frac{u^4}{16} - \cdots\right]_1^0$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}\,.$$

We can do the same to our second integral, rewriting our integration as two summations:

$$\int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k}\, du = \sum_{k=1}^{\infty} \int_0^1 \left(1 + \frac{u}{2} + \frac{u^2}{3} + \frac{u^3}{4} + \cdots\right) du$$
$$= \sum_{k=1}^{\infty} \left[u + \frac{u^2}{4} + \frac{u^3}{9} + \frac{u^4}{16} + \cdots\right]_0^1$$
$$= \sum_{k=1}^{\infty} \frac{1}{k^2}\,.$$

Therefore,

$$\lim_{b\to\infty} \int_0^b \ln\left(\frac{e^x+1}{e^x-1}\right) dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k^2}\,.$$

## 5 Using the Riemann zeta function and the Dirichlet eta function to find sums

In order to find these sums, we can use the Riemann zeta function; see [4]. The Riemann zeta function, denoted as $\zeta(n)$, can be defined as follows:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}\,.$$

For our purposes, the particular value of the function, $\zeta(2)$, discovered by Leonhard Euler (see [5]), is useful as it can be substituted for part of our equation above:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}\,.$$

While the numerous proofs for this are quite extensive, and not exactly necessary for our purposes, I would encourage you to explore them on your own. The article [6] has a nice collection of proofs that are fascinating!. Another function, the Dirichlet eta function, denoted as $\eta(n)$, is the alternating form of the Riemann zeta function:

$$\eta(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n}\,.$$

Like the Riemann zeta function, the Dirichlet eta function has particular values; see [7]. You'll notice above that one summation is just the alternating form of the other, which is represented by the Riemann zeta function. The second part of our overall sum can therefore be represented by the Dirichlet eta function of 2:

$$\eta(2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}\,.$$

Now our job is simple: we just need to substitute these values in and add them together. Observe that

$$\int_0^1 \left(\frac{1}{u}\right) \ln(1+u)\,du - \int_0^1 \left(\frac{1}{u}\right) \ln(1-u)\,du$$

is equal to

$$\int_1^0 \sum_{k=1}^{\infty} \frac{(-1)^k u^{k-1}}{k}\,du + \int_0^1 \sum_{k=1}^{\infty} \frac{u^{k-1}}{k}\,du$$

and thus to

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{\pi^2}{4}\,.$$

Despite the graph indicating that the area under our function was seemingly infinite, the use of a couple integration methods and important mathematical functions has shown us that our integration is in fact finite.

## Acknowledgements

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