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A pretty accurate solution to the Delian problem

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1 Context

The doubling of the cube, also known as the *Delian problem*, is one of the three ancient problems from the 5th century BC. The two others are "Squaring the circle" [1] and "Trisecting the angle".

The term "Delian problem" is based on a story in which a serious plague broke out on the Greek island of Delos[3]. The islanders sought advice from the oracle at Delphi, asking what they could do to improve the situation. The oracle requested that the cube-shaped altar in the island's Temple of Apollo be enlarged so that the new cube-shaped altar would have twice the volume of the original. This essentially amounts to the construction of a line segment of length $\sqrt[3]{2}$.

According to Plutarch [2, p. 4], when the Delian architects questioned Plato on how to carry out the task, he referred them to Archytas of Tarentum, Eudoxus of Cnidos and Menaichmos. Each of the three respected mathematicians found one or two (exact) solutions: Archytas found the required length in the form of a curve resulting from the intersection of a cylinder, a horn torus and a cone section; Eudoxos constructed the so-called "two mean proportionals" using a curve; and Menaichmos found two other solutions. The first solution intersection a parabola and a hyperbola, and the second intersected two parabolas. [3, p. 4]. Though none of the above solutions can be constructed on a plane with the simple tools of an architect: a compass and a straightedge.

In 1837, Pierre Wantzel [4] proved that only lengths which are algebraic numbers can be constructed with compass and straightedge. More precisely, the constructed numbers turn out to be all the numbers that you can calculate in a finite number of steps using the four arithmetic operations $+$, $-$, $*$, $/$ as well as taking square roots $\sqrt{}$. This is the reason why we can construct for example $\sqrt{2}$ which is the diagonal of a 1×1 square $\square$, or $\sqrt{5}$ which is the diagonal of a *"Quadratum Lungum"*, a 1×2 rectangle ▭. Since $\sqrt[3]{2}$ is not an algebraic number, it is now definitely proven that doubling the volume of a cube, using solely a compass and straightedge, is impossible.

Since Wantzel's discovery, mathematicians have tried to construct the best possible geometric approximations of $\sqrt[3]{2}$ using solely a compass and straightedge. One should also keep in mind the rules of geometrography [5, p.17]: the greatest accuracy is physically achieved with the construction requiring the fewest construction steps.

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[3] Delos is part of the Cyclades in the Aegean sea and played a major role in antiquity since it is the mythical birthplace of the sun god Apollo.

2 Rules of construction with compass and straightedge

The rules of construction with "compass and straightedge" are the following:

- R1 We are initially given just two points.
  The distance between these points defines 1 unit.
- R2 We are allowed to draw a straight line through any two given points.
- R3 We are allowed to draw a circle with centre in a given point that intersects another given point.
- R4 Each intersection of a drawn line or circle with another line or circle defines a new given point.
- R5 We are allowed only finitely many steps.

A selection of basic constructions like building a parallel line through any given point (5 steps), or transposing a line segment (11 steps), are illustrated in the article [1].

3 Doubling the cube, an algebraic approach

There are different ways to tackle this ancient problem. In this paper, we will present two very different approaches leading to interesting results. Let's start with a pretty good algebraic approximation and then work out a geometric figure to suit.

The algebraic approximation

$$\sqrt[3]{2} \approx \frac{10}{3\sqrt{7}}$$

is correct to 0.003% accuracy. The simple expression suggests a simple construction.

We start by drawing the line through the two given points A and B which define 1 unit of length (1 step), then five circles define the points C,D,E,F and G so that $|\mathrm{AC}| = 2$, $|\mathrm{BD}| = 3$, $|\mathrm{BE}| = 5$, $|\mathrm{DF}| = 6$ and $|\mathrm{BG}| = 10$ (5 steps). The intersection of the two circles with radius $|\mathrm{DF}|$ respectively centred at D and F gets us the point R (2 steps). Since DFR is an equilateral triangle of side length 6, its undrawn height is $|\mathrm{RH}| = 3\sqrt{3}$. It follows that in the right-angled triangle BHR,

$$|\mathrm{BR}| = \sqrt{|\mathrm{BH}|^2 + |\mathrm{HR}|^2} = \sqrt{6^2 + (3\sqrt{3})^2} = \sqrt{63} = 3\sqrt{7}\,.$$

We find the point S on the line through B and R so that $|\mathrm{BS}| = 1$ (1 step) and we construct a line parallel to GR running through S to get point T (5 steps). Since $|\mathrm{BS}| = 1$, the triangle similarity of BST and BRG implies that

$$|\mathrm{BT}| = \frac{|\mathrm{BT}|}{|\mathrm{BS}|} = \frac{|\mathrm{BG}|}{|\mathrm{BR}|} = \frac{10}{3\sqrt{7}} \approx \sqrt[3]{2}\,.$$

Since we construct this length in 14 steps only, this drastically limits the incremental error due to hand-drawing. Therefore, the proposed figure is likely to be the *geometrographic* solution of the Delian Problem. Though, one could argue that the Delian architects could not benefit from the beauty of modern algebra.

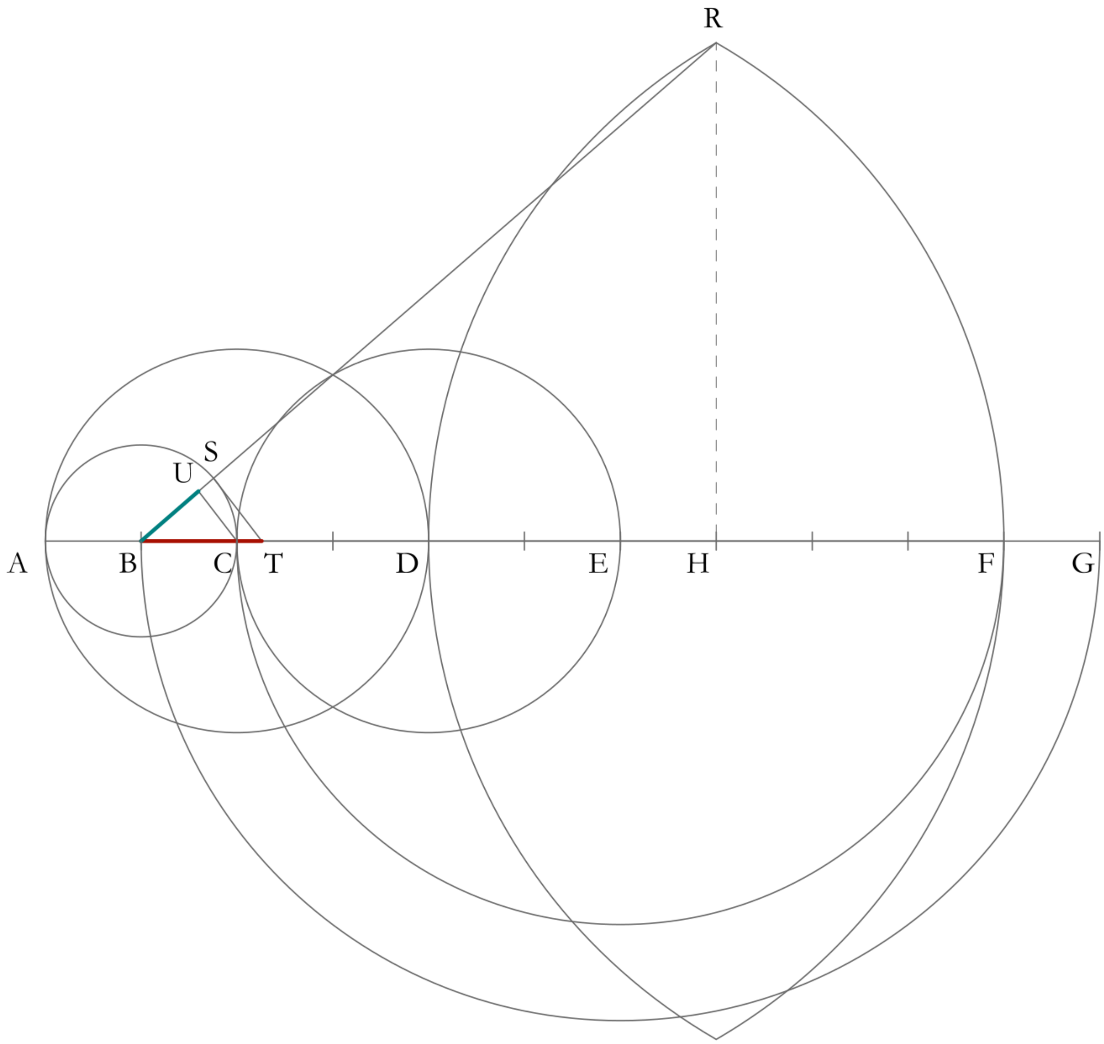


Figure 1: Constructing $|\mathrm{BT}| = \frac{10}{3\sqrt{7}} \approx \sqrt[3]{2}$ and $|\mathrm{BU}| = \frac{3\sqrt{7}}{10} \approx \frac{1}{\sqrt[3]{2}}$

Should we aim to construct a cube with half the volume of the original, then we can follow the very same approach and construct a line parallel to GR running through C to get point U. Since $|\mathrm{BC}| = 1$, the triangle similarity of BCU and BGR implies that

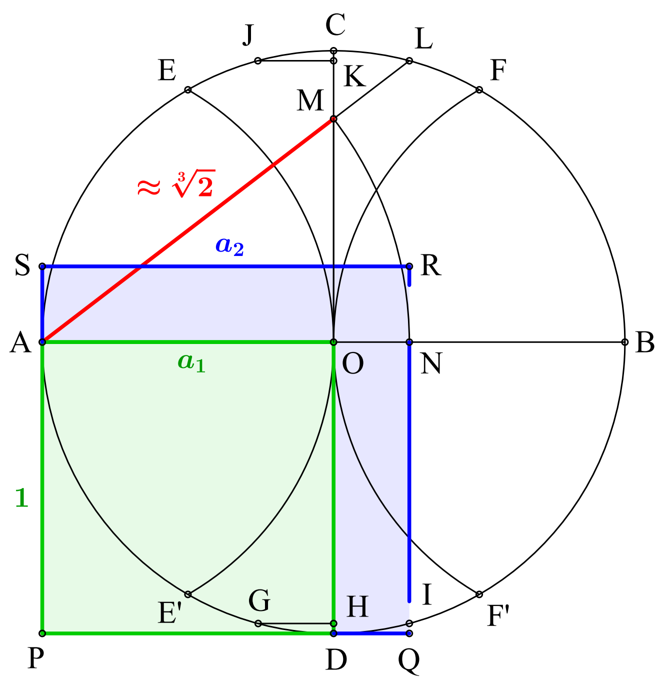
$$|\mathrm{BU}| = \frac{|\mathrm{BU}|}{|\mathrm{BC}|} = \frac{|\mathrm{BR}|}{|\mathrm{BG}|} = \frac{3\sqrt{7}}{10} \approx \frac{1}{\sqrt[3]{2}}\,.$$

Thus, a cube with edge length $|\mathrm{BU}|$ has very nearly half the volume of the original cube.

4 Doubling the cube, a geometric approach

If we maintain the geometrical spirit of ancient Greek architects, then we can work hard using solely the tools of geometry ...and a lot of perseverance. Here is a pretty accurate solution which has also the merit of being simple and elegant. Here are the construction steps.

It starts with the unit circle with radius $|OA| = 1 = a_1$ and the drawing of the diameter AB. Next, the centre line CD is drawn perpendicular to the diameter AB. This is followed by an arc of radius $|OA|$ around A and B, respectively; the intersections are E, E' and F, F'. An undrawn perpendicular bisector of length $|E'D|$ bisects the arc $OE'D$ in G. A line parallel to AB through G gives the line segment GH. The point I is determined with the help of an undrawn perpendicular bisector of length $|HF'|$.



Now transpose the line segment IF' onto the circular arc OCE from the point E; this gives the intersection J. A line parallel to AB through J gives the line segment JK. The point L is determined with the help of an undrawn perpendicular bisector of length $|KF|$. The line segment AL intersects OC in M and thus yields the line segment AM whose length is very nearly equal to $\sqrt[3]{2}$. The calculated error is $2.375\ 10^{-16}$.

To put this error in perspective, consider a cube with edge length $a_1 =$ *1 billion km*.[4] The constructed edge length a_2 of the doubled cube would be wrong by 0.2 millimeters.

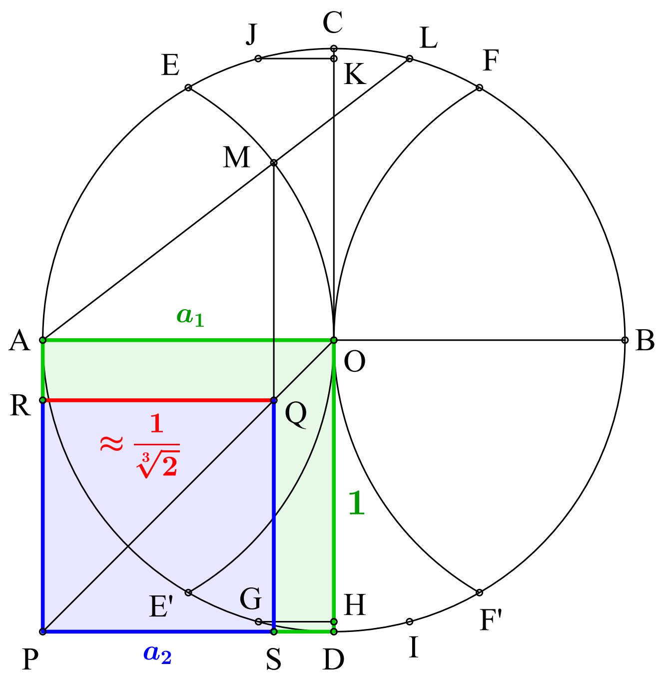
[4] You could fit 78480 copies of planet Earth side by side on this length. Light would take about 56 minutes to cover this distance.

5 Bonus: Halving the cube

By a very similar construction, we can also halve the cube to a similar level of accuracy. You will note that both constructions are the same except for the following step:

The point L is determined with the help of an undrawn perpendicular bisector of length $|KF|$.

We then continue with the following step: the line segment AL intersects the circular arc AE′E in M. The construction of the square PDOA with side length a_1 and diagonal OP follow. Finally, the parallel MQ to CD yields the cosine of the angle OAM, equal to the line length |QR| which is very nearly equal to $\frac{1}{\sqrt[3]{2}}$. The calculated error is $2.6\ 10^{-16}$.



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