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Application of Lévy stable distributions to stock market returns using econophysics

James Houghton[1] and **Kristin Osika**[2]

1 Introduction

The ability to forecast potential fluctuations of asset prices is critical to maintaining the stability of the world economy. When asset prices decline greatly without warning, banks, pension funds, consumer savings, and in some cases even governments themselves are exposed to significant losses. Research to improve models that capture stock price volatility has been a major focus of economists ever since Eugene Fama published the random walk theory for describing market prices in 1965. In the last two decades, research in the field of econophysics, the intersection of economics with quantitative techniques from physics, has generated models that greatly improved the predictability of asset price fluctuations through historical data. One such approach, using Lévy Stable distributions, has been demonstrated to have a high predictive ability of extreme market events, including the financial crisis of 2008. This paper first will investigate Lévy Stable distributions, how they can be fit to stock market return data, and the techniques utilized to convert these distributions into predictive indicators of stock market crashes. Additional analysis extends recent research to the 2020 stock market crash to further investigate the predictive ability of the Lévy distribution parameters. The results demonstrate that the Lévy distribution parameters changed significantly before the major part of the 2020 stock market crash, confirming their value as a predictive indicator of extreme market events.

2 History

The history of financial markets is rich with stories of boom and bust cycles, instances where markets depart from "normal" behaviour during cycles of high volatility. The consequences can be disastrous, from wiped out retirement savings to bank failures and even periods of recession. The practice of market analysis is of critical importance to understanding financial risks and protecting individuals, financial institutions, governments, and the financial system itself from ruin. Attempting to predict extreme market events in advance may prove to be challenging, but establishing a framework for quantifying the likelihood of stock market movements is of utmost importance to maintaining the stability of critical financial market banking institutions.

[1]James Houghton is a final-year student at Pingry School, New Jersey, USA
[2]Kristin Osika is a final-year student at Pingry School, New Jersey, USA

Early methods of stock market analysis

Prior to the 1960s, stock market analysis was largely conducted within two main frameworks: technical and fundamental analysis. Technical analysis is based on the belief that historical patterns repeat themselves [1]. Technical analysts first develop a familiarity with different types of historical price fluctuations and attempt to identify instances of repeating patterns early in their development. From a statistical standpoint, this method incorporates an assumption of price dependence whereby subsequent prices are determined by preceding patterns [1]. Fundamental analysis, in contrast, is based on the assumption that all securities have an intrinsic value based on their earning potential. Fundamental analysts attempt to estimate the value of securities based on financial statements, competitors, and the market, and also assume that the prices of securities trend towards their intrinsic value [2].

During the modern digital age in the mid 20th century, research enabled by developments in computing power demonstrated serial correlation among stock prices to be near zero and thus established the independence of stock market returns from their historical performance [3]. This result contradicts the core assumption of charting analysis which assumes that prices follow patterns based on recent fluctuations.

Subsequently, economist Eugene Fama popularized an alternative theory for the movement of stock-market prices in his 1965 paper "Random Walks in Stock-Market Prices" [3]. The random walk theory first assumes that all available information is known by market participants and thus that the current values of stocks are approximately equal to their intrinsic values [3]. Subsequent price fluctuations are then determined by the introduction of new information which is immediately incorporated into security prices [3]. Under this framework, new information is unpredictable and random, and security prices behave randomly and have no dependence on their historical values [3]. Fundamental analysis remains important to stock market analysis under the random walks model, as those analysts who are more skilled at assessing the true intrinsic value of a security relative to the market as a whole will likely have more investment success.

Development of portfolio theory

Succeeding work of economists focused on developing portfolio strategies to maximize profit for a certain amount of risk. Nearly all of the research leading up into the early 1990s relied on the core assumption that stock market returns followed a Gaussian normal distribution [4]. The normal distribution has many convenient characteristics lending to a simplicity of calculations, such as the sum of multiple normal distributions remaining normal, with established formulae for the resulting mean and variance. These calculational efficiencies led to a series of important breakthroughs including modern portfolio theory, the capital asset pricing model, and the Black-Scholes option pricing theory that revolutionized investing and risk management during the second half of the 20th century [4, 5]. However, the normal distribution, underpinning the calculations for all of these techniques, significantly underestimates the probabilities of extreme market events [6]. In other words, the tails of the normal distribution are not fat enough to accurately estimate the frequency of large fluctuations in price movements in the real world. This underestimation introduces risk-management challenges, and as the prevailing theories do not assign accurate likelihoods to large movements in asset prices, financial institutions become exposed to greater levels of risk of unpredictable instances of failure. The traditional quantification of stock market variation has fallen short in this area, leaving the investigation of an important area of research incomplete for decades.

The emergence of econophysics and the Lévy stable distribution

At the end of the 20th century, physicists began to focus research efforts on stock prices and the economy, together inventing the field of econophysics. Econophysics was first introduced in a paper by Stanley *et al.* [7] in 1996. The term econophysics generally applies to any investigation of economic problems by physicists [8]. The main objective of the field is to parameterize stock price movements by applying models from statistical physics. Physicists have investigated a large number of diverse frameworks to model the stock market return's distribution function [8]. One approach, called quantum finance, translates particle motion into wave functions to represent stock price movements [9], and another proposes modeling stock price movements as energy releases, which cause prices to change as if traversing an electrostatic field [10]. Econophysicists have even investigated the question of whether or not the market has "memory" and how that could influence returns [11]. Finally, the Lévy Stable distribution, which will be discussed in this paper, has been shown by numerous academic research papers to fit the empirical data of stock market returns well, unlike the Gaussian normal distribution under which much of modern finance theory is based.

The Lévy distribution was first researched by the Italian economist Vilfredo Pareto during his analysis of the distribution of income in the early 20th century [12]. French mathematician Paul Lévy greatly expanded the research of the distribution in the 1920s while investigating how identically distributed independent random variables behave when added together [13]. A defining characteristic of the Lévy distribution is that a sum of random variables distributed according to a Lévy distribution is itself a Lévy distribution [13]. One major inconvenience of the Lévy distribution lies in the fact that its probability density function, the traditional means of analysing a probability distribution, is not able to be expressed in a closed-form formula [13]. Despite the lack of a closed-form probability density function, physicists were able to analyze the distribution's applicability to stock market and other data sets using computationally intensive methods aided by computers. Recent research has demonstrated that changes in values of the Lévy distribution parameters are extremely predictive of market crashes, including the Great Depression, financial crisis of 2007-2008, and 1987 crash [14, 15]. In this paper, Lévy distributions will be compared to the standard normal distribution, applied to a large period of Standard and Poor's 500 (S&P 500) Index returns, and be assessed for their predictive ability of the 2020 stock market crash after the onset of COVID-19.

3 The Lévy stable distribution and methodology

The Lévy Stable Distribution

All Lévy distributions can be described by four parameters:

$$\begin{aligned} \textit{stability} &\ \alpha \in (0,2] \\ \textit{skewness} &\ \beta \in [-1,1] \\ \textit{scale} &\ \gamma \in [0,\infty) \\ \textit{location} &\ \delta \in (-\infty,\infty). \end{aligned}$$

The Lévy distribution has a probability density function

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(k) e^{ixk} dk \tag{1}$$

where

$$\phi(k) = e^{i\delta k - |\gamma k|^{\alpha}\left(1 + i\beta\,\mathrm{sgn}(k)\omega(\alpha,k)\right)} \tag{2}$$

and

$$\omega(k,\alpha) = \begin{cases} \tan\frac{\pi\alpha}{2} & \alpha \neq 1 \\ \frac{2}{\pi}\ln|k| & \alpha = 1\,. \end{cases} \tag{3}$$

As the integral in (1) cannot be evaluated without using numerical integration, there is no simpler expression for the probability density function. Special cases of the Lévy distribution with defined parameters include the Cauchy distribution where $\alpha = 1$ and $\beta = 0$ and the Gaussian normal distribution $\alpha = 2$ and $\beta = 0$ [14]. Note that k is not a parameter of the Lévy distribution itself, but does play a key role in the description of the density function through $\phi(k)$, which is referred to as the characteristic equation.

Comparing the Lévy distribution to the normal distribution

By comparing the probability density functions of the Gaussian normal distribution to Lévy distributions with various factors, the key differences in the distributions are able to be examined. It is not possible to easily graph the shape of a Lévy distribution due to the complexity of its probability density function. Numerical integration methods are required to produce the probability of each individual point along the x-axis in the interval $(-\infty,\infty)$. An alternative approach, as used in this paper, is to use a computer to generate a large number of random samples from a Lévy distribution, and then convert the data set into a probability density function graph. Larger sample sizes produce smoother graphs at the expense of increased computational resources.

In order to make a comparison graph, 100,000 sampled random values from the normal distribution and Lévy distribution were generated. The programming language R was used to generate the values. The `rnorm()` function in the `stats` package, maintained by R Core Team, was used to generate the values from the normal distribution [16]. The `rstable()` function in the `stabledist` package was used to generate the values from the Lévy distribution [17]. Figure 1 displays these values as generated probability density functions for both the standard normal and Lévy distributions, where $\alpha = 1.6$, $\beta = 0$, $\gamma = 1$, $\delta = 0$. The stability parameter α was set to 1.6 as it typically is the result when fitting a Lévy distribution to stock market returns. The skewness β and mean δ were set to 0 and the scale factor γ was set to 1 to enable a direct comparison to the normal distribution with mean zero and standard deviation 1.

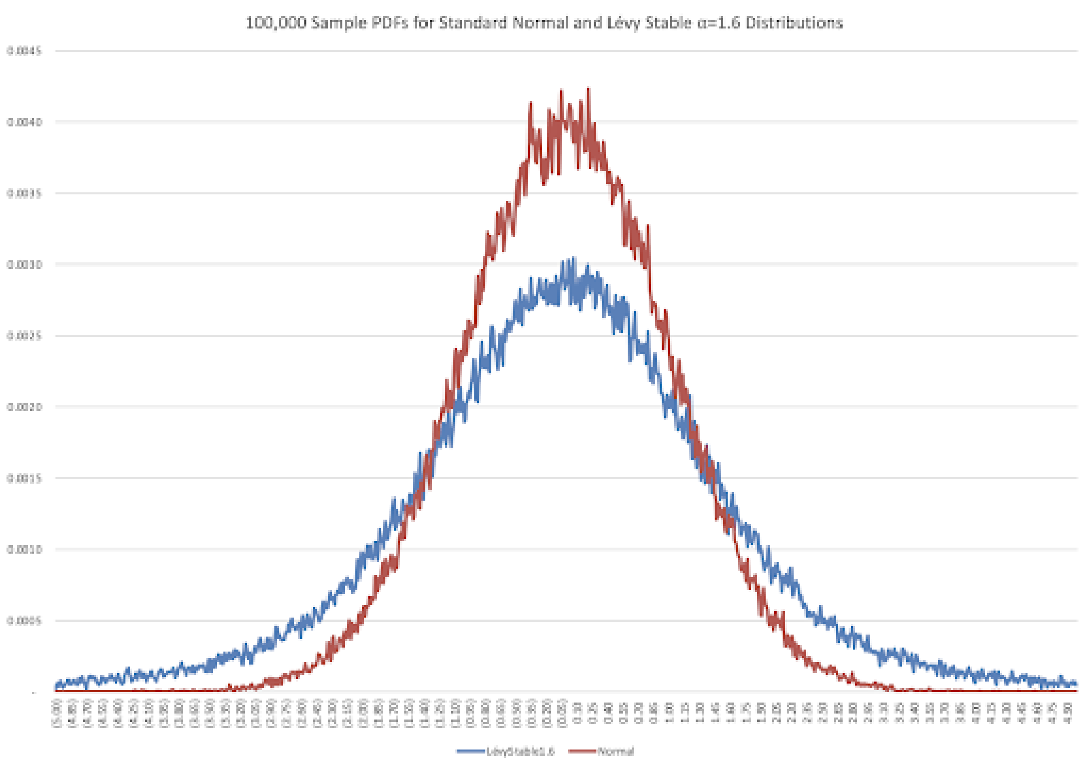

Figure 1: 100,000 sample probability density functions for standard normal and stable $\alpha = 1.6$ distributions.

Fitting a Lévy stable distribution to S&P 500 stock market returns

Analysis of historical stock index returns is typically conducted by economists using logarithmic returns rather than typical percentage returns [6]. Logarithmic returns have the advantage of uniform measurement of price movement in both the positive and negative direction [18]. For example, when a stock moves from $100 to $125 and then back to $100, percentage returns measure that movement as +25% and -20%, while logarithmic returns are the same at $\pm 22.3\%$.

Once a series of price data is collected, daily logarithmic returns, r_t, can be calculated by subtracting the logarithms of successive prices, where p_t equals the stock price on day t:

$$r_t = \ln p_t - lnp_{t-1} = \ln \frac{p_t}{p_{t-1}}\,. \tag{4}$$

The daily percentage return on day t is thus equal to $e^{r_t} - 1$.

In order to determine the particular Lévy distribution that provides the best description of stock returns, a method that estimates the distribution parameters based on a time series of logarithmic return information is needed. One approach, which is used in this paper, was developed by Ioannis Koutrouvelis in 1980. The Koutrouvelis algorithm entails rearranging the characteristic function $\phi(k)$ defined in (2) of the Lévy distribution into the following equations [13, 19]:

$$\ln\big(-\ln|\phi(k)|^2\big) = \ln(2\gamma^\alpha) + \alpha \ln|k| \tag{5}$$

and

$$\arctan\frac{\operatorname{Im}\phi(k)}{\operatorname{Re}\phi(k)} = \gamma k + \beta\gamma^\alpha\Big(\tan\frac{\pi\alpha}{2}\Big)\operatorname{sgn}(k)|k|^\alpha \tag{6}$$

where

$$\operatorname{Im}\phi(k) = e^{-|\gamma k|^\alpha}\sin\Big(\delta k + |\gamma k|^\alpha\beta\operatorname{sgn}(k)\tan\frac{\pi\alpha}{2}\Big) \tag{7}$$

and

$$\operatorname{Re}\phi(k) = e^{-|\gamma k|^\alpha}\cos\Big(\delta k + |\gamma k|^\alpha\beta\operatorname{sgn}(k)\tan\frac{\pi\alpha}{2}\Big) \tag{8}$$

where $\operatorname{Im}\phi(k)$ and $\operatorname{Re}\phi(k)$ are the imaginary and real parts of (2).

In order to estimate the parameters, $\phi(k)$ is calculated based on an initial guess for each of the four parameters and the observed stock market returns for a range of values of k. (5) is then used to run a two-factor regression to determine the first iteration of values for α and γ. Next, (6) is used to perform a second two-factor regression to determine the first iteration of values for β and δ. The initial iteration of distribution parameters is then tested against the observed stock market returns. If the distribution does not fit the observed data within a desired tolerance, then the procedure is repeated with the newly determined parameters as the next initial guess. Once satisfactory values for the parameters are reached, the best fit Lévy distribution for the stock market return data has been determined [19].

Lévy distribution parameters that model the S&P 500, which represents a significant portion of the overall market capitalization of the stock market in the United States, were calculated using the Koutrouvelis method. Closing values of the S&P 500 Index were obtained from December 31, 1999 to November 30, 2021 from Yahoo Finance [20]. A time series of daily logarithmic returns r_t was created for t from January 3, 2000 to November 30, 2021. There are 5,514 data points in the return time series representing approximately 21 trading days per month over the 263 month period to be tested. The `stableEstim` package within the R programming language includes the Koutrouvelis iterative regression algorithm and was used to determine the parameter

Lévy Parameter	Koutrouvelis Regression Result
α	1.536138
β	-0.311556
γ	0.005644246
δ	0.001723908

Table 1: Koutrouvelis Regression Parameter Values

values for a Lévy distribution against the S&P 500 data [21]. After uploading the data into R-Studio, the Koutrouvelis regression was executed with the results in Table 3.

The α parameter of 1.54 is consistent with the expected result of 1.6, and the β parameter of -0.312 indicates a negatively skewed distribution, consistent with overall average positive returns exhibited by the S&P 500 index during the observation period. In order to further assess fit, a randomly generated sample of 100,000 Lévy distribution points using the S&P 500 Lévy parameter results were generated to graph an approximate probability density function against the probability density curve of the observed S&P 500 Index returns; see Figure 3. The overall peak of the empirical S&P 500 returns is slightly higher than the fitted Lévy distribution but, importantly, the tails of the Lévy distribution show a strong alignment with the S&P 500 data, indicating a good fit with the actual probability of extreme market events.

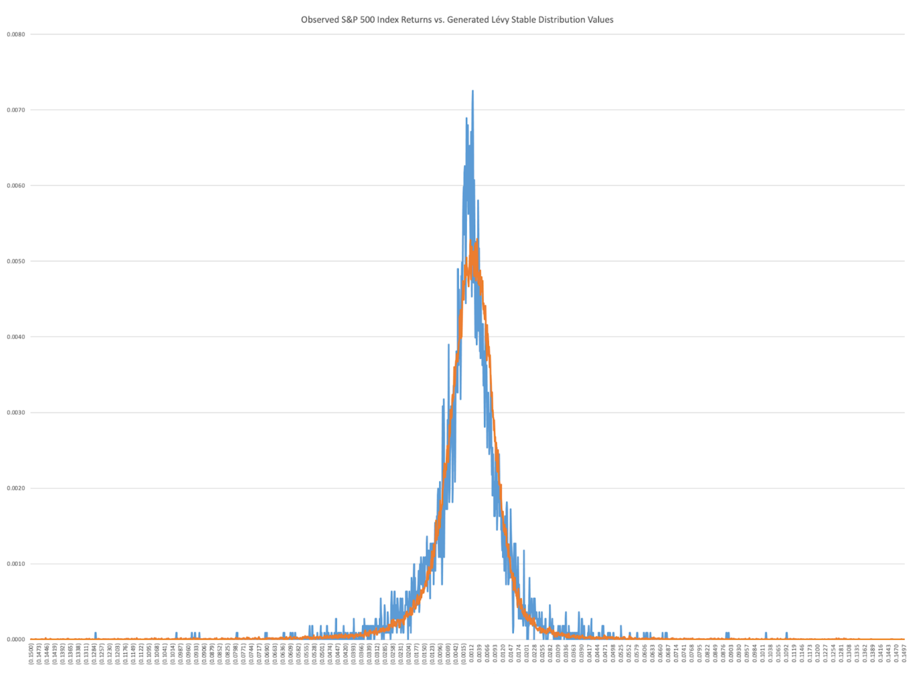

Figure 2: Observed daily S&P 500 Index returns between January 3, 2000 and November 30, 2021 and 100,000 generated Lévy Stable Distribution Values using the parameter values in Table 1.

4 Results and analysis

Work recently published by Fukunaga [14] and Bielinskyi [15] has investigated the idea that the distribution of stock market returns is inherently unstable itself and the potential distribution of returns changes daily. In order to determine how the Lévy distribution parameters change over time, the parameters can be calculated based on the historical stock market returns for a fixed number of days leading up to and ending on that day. A short window would be more responsive to changes in market conditions, but a series too short will not contain enough market returns to accurately fit the distribution. The best fit Lévy parameters calculated for each day can then be observed over time by graphing the parameters for each day.

Both Fukunaga and Bielinskyi conducted such an analysis and presented findings in published work in 2018 [14] and 2019 [15]. Bielinskyi's work in particular indicates that both the α and β parameters have the potential to predict stock market crashes. Bielinskyi calculated and graphed the α (left) and β (right) Lévy parameters from the 1987 stock market crash through 2018 using a window of the previous 500 trading days. A graph of the parameter values against the Dow Jones Industrial Average can be seen in Figure 3 [15]:

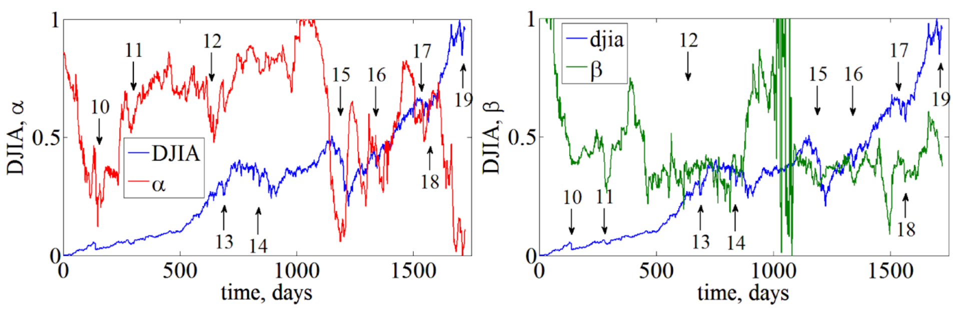

Figure 3: Fitted Lévy parameters α (left) and β (right) against the Dow Jones Industrial Average from the 1987 stock market crash through 2018 [15].

As Bielinskyi noted, α declines and β trends towards zero when the market is leading up to a crash event. For example, Figure 4 depicts α declining dramatically well in advance of the Dow Jones Industrial Average crashing during the financial crisis of 2008 indicated by arrow 15. The parameter β also seems to trend to zero just in advance of the 2008 financial crisis, but the movement is less significant.

In this paper using the same Koutrouvelis method, the α and β parameter values will now be calculated for the more recent time frame including the 2020 stock market crash. However, a 200 day window instead of a 500 day window was chosen for this analysis in order to potentially increase sensitivity to instability in the market. Lévy parameter values for α and β were generated from July 1, 2019 through June 30, 2020 and can be seen in Figure 4 and 5, respectively, along with the S&P 500.

As depicted in Figure 4, α declines sharply along with the S&P 500 during the 2020 market crash. However, in contrast to historical stock market crashes, there is no

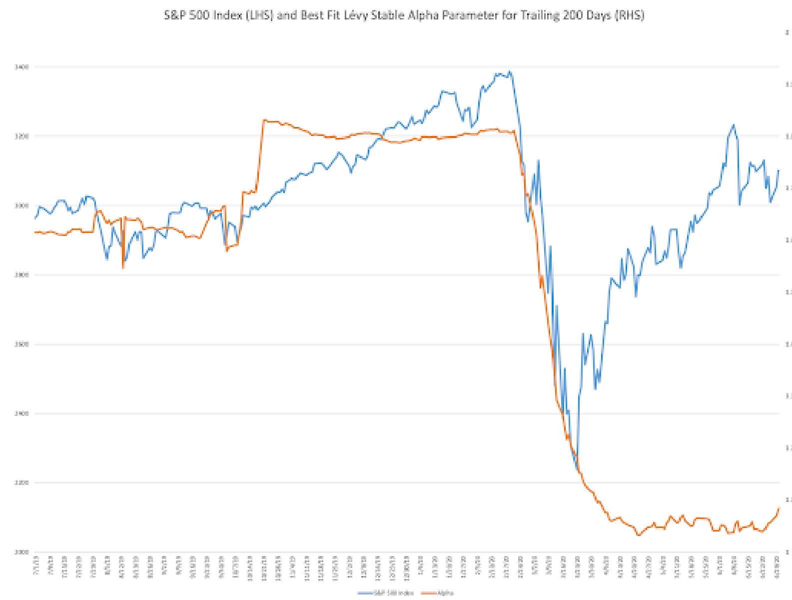

Figure 4: Fitted Lévy α parameter against the S&P 500 Index from July 1, 2019 through June 30, 2020.

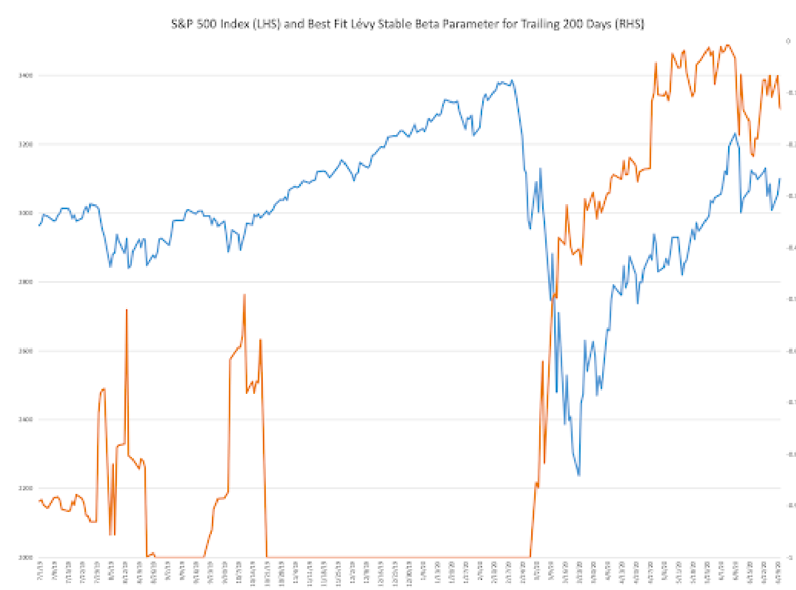

Figure 5: Fitted Lévy β parameter against the S&P 500 Index from July 1, 2019 through June 30, 2020.

evidence of α declining before the very beginning of the 2020 market crash, only just after. The parameter does show a strong further decline even as the market makes an initial rebound before a more pronounced crash in the middle of March. This means that the main component of the market crash from March 4, 2020 to March 24, 2020 is still well predicted by a decline in α in advance.

In addition, as shown in Figure 5, the β parameter shows strong movement towards 0, indicating a less negatively skewed distribution of returns as the S&P 500 became more volatile throughout the market crash and subsequent rebound. Overall, the α and β parameters were good leading indicators of the main crash, and could have been used as signals of the greater market crash during the initial rebound.

5 Conclusions

The contributions of physicists to the broader topic of economic research have greatly advanced the overall understanding of market behaviour. Through research of the Lévy distribution, econophysics has vastly improved the understanding of tail behaviour in markets, which had received less focus despite its critical impact on risk management, as the main source of economic risk lies in the tails. The distribution's parameters clearly have great potential as indicators of stock market crashes and further study of their relationship to market crash events is warranted. As shown in previous research for historical stock market crashes and in this paper for the COVID-19 crash, the Lévy distribution parameters provided a signal before the market's decline. However, unlike historical events, α and β were not able to predict the initial market decline of the 2020 stock market crash. Perhaps the nature of COVID-19, a catalyst completely external to the market, reduced the predictive value of the Lévy parameters in this instance. A deeper understanding of market signals available through this methodology can potentially be of great value to investors, financial institutions, and governments seeking to better manage risk and protect the stability and soundness of the broader financial system.

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