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Problems 1701–1710

Parabola would like to thank Toyesh Prakash Sharma for contributing Problem 1710.

Q1701 A school class consists entirely of twins: $2n^2 + 2n$ pairs of them, where $n \geq 2$. Including the teacher, there are $4n^2 + 4n + 1$ people in the class, so they can stand in a $2n+1$ by $2n+1$ square array. Prove that however they arrange themselves in this array, it will be possible to find $2n + 1$ of the children (excluding the teacher) in such a way that no two of the chosen children are standing in the same row, no two are standing in the same column, and no two are twins.

Q1702 Consider the sequence of numbers obtained by stringing together the digits of the positive integers, namely

$$1\,,\ 12\,,\ 123\,,\ 1234\,,\ 12345\,,\ 123456\,,\ldots$$
$$\ldots,\ 12345678910\,,\ 1234567891011\,,\ 123456789101112\,,\ldots$$

Are any of these numbers multiples of 11? If so, then find the smallest such multiple.

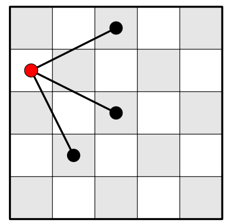
Q1703 Solve the simultaneous equations

$$x\sqrt{x} + y\sqrt{y} = 183 \qquad \text{and} \qquad x\sqrt{y} + y\sqrt{x} = 182\,.$$

Q1704 The Chinese red packet is a tradition that goes back over 2000 years as a way of gifting money to children for the Lunar New Year, and also at weddings. The problem below requires a blend of luck and skill.

A player is faced with 1001 red packets of which 5 lucky packets contain money (and the other 996 are empty). In each round, the player selects any number of the envelopes and divides them into a maximum of 5 piles. The player is then told the number of lucky packets in each pile. The objective is to obtain 5 piles of red packets where each pile has exactly one lucky packet. What is the maximum number of rounds required?

Q1705 In the game of chess, a knight can move from its current square to any square reached by moving horizontally or vertically by two squares, then by one square in a perpendicular direction. The diagram



shows all the moves available on a 5×5 chessboard to a knight on the white square marked with a red dot. Since a single move takes a knight from a white square to a black square, or *vice versa*, no two of the 13 black squares are separated by a single knight's move. Is there any other choice of 13 squares on a 5×5 board for which this is true?

Q1706 Arrange a hundred digits (digits are $0, 1, \ldots, 9$) on a circle in such a way that, reading clockwise, every one of the pairs $00, 01, \ldots, 99$ occurs once each.

Q1707 Consider all numbers that are the product of eleven different positive integers whose sum is 82. For example, one of these products is

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 27\,.$$

Find the greatest common divisor of all of these products.

Q1708 How many paths of length $m+n+2$ are there from $(0,0)$ to (m,n) on an $m \times n$ grid, if the path may never visit the same grid point more than once?

Q1709 Find the largest example of an 8–digit number which uses each of the digits $1, 2, 3, 4, 5, 6, 7, 8$ just once and which is a multiple of 101.

Q1710 Suppose that a, b, c are positive real numbers whose sum is three: $a+b+c = 3$. Prove that

$$\frac{a^2+b^2}{\sqrt{ab}} + \frac{b^2+c^2}{\sqrt{bc}} + \frac{c^2+a^2}{\sqrt{ca}} \geq 6\,.$$