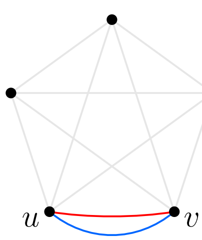
Parabola Volume 59, Issue 1 (2023)

Solutions 1691–1700

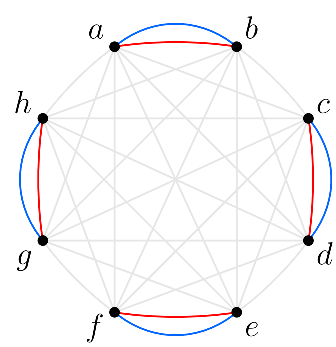
Q1691 This problem has been modified in order to take advantage of a fine solution contributed by Nye Taylor[1]. The problem set in the last issue consisted of part (b) only.

(a) Five points are drawn on a page. Two points u and v are joined by both a red curve and a blue curve. All other pairs are joined by one line (or curve) which is shown in the diagram as grey, and will be coloured either red or blue.



Prove that, no matter how this colouring is done, the resulting diagram will contain three of the five original points mutually joined by red lines, or three points mutually joined by blue lines.

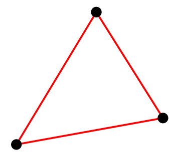
(b) Eight points a, b, c, d, e, f, g, h are drawn on a page. Four pairs are joined by red and blue curves, as shown in the diagram. All other pairs are joined by one line, which will be coloured either red or blue.

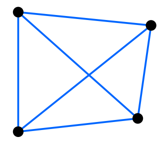


Prove that, no matter how this colouring is done, the resulting configuration will contain three points mutually joined by red lines, or four points mutually joined

[1]Nye Taylor is a student at UNSW Sydney.

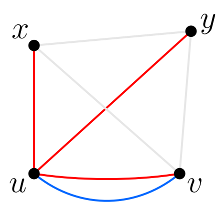
by blue lines.





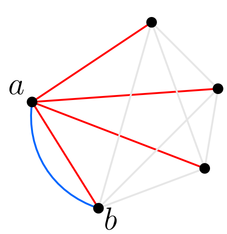
SOLUTION Terminology: we will refer to the connections between points as "lines" (even if they are curved!). We will call the sought–for configurations "a red triangle", "a blue triangle" and "a blue quadrilateral", and we will say that two points joined by a red (or blue) line are "red–neighbours" (or "blue–neighbours"). Points joined by two lines will be called "partners".

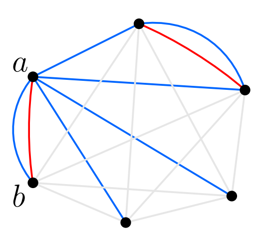
To prove (a), consider the point u. Excluding its partner v, each of the other three points is either a red–neighbour or a blue–neighbour of u; since there are three points here and only two options, it must be that either two or more of the three points are red–neighbours of u, or two or more are blue–neighbours of u. Suppose there are two red–neighbours, and call them x and y. Then u has three red–neighbours: its partner v,



and the points x and y just mentioned. If any two of v, x, y are connected by a red line, then these two, together with u, form a red triangle; if not, then v, x, y form a blue triangle. If u were to have two blue–neighbours rather than two red–neighbours, then a nearly identical argument would show that the configuration contains a blue triangle or a red triangle. This completes the proof for part (a).

Now for part (b), consider the point a. Excluding its partner b, each of the other six points is either a red–neighbour or a blue–neighbour of a; so there must be either 3 or more red–neighbours, or 4 or more blue–neighbours. Consider the former case: then (restoring b) the point a has 4 red–neighbours; see the first diagram below. Either two of these are joined by a red line, making a red triangle together with a; or all are joined by blue lines, making a blue quadrilateral.





In the remaining case, a has (at least) 4 blue–neighbours among the points c, d and e, f and g, h. Since we have 4 blue–neighbours chosen from 3 pairs, two of them must belong to the same pair and therefore must be joined by both a red line and a blue line. Now restore b; see the second diagram above. Then the blue–neighbours of a include 5 points with at least one pair of points joined by lines of both colours. By part (a), there are two options: either these five points include a red triangle; or they contain a blue triangle, which together with a makes a blue quadrilateral.

We have considered every possibility, and have shown that no matter what the colouring, our configuration must contain a red triangle or a blue quadrilateral. This concludes the proof of (b).

Q1692 Prove that the sum of two different powers of 2 can never be a cube or higher power of an integer. That is, there are no solutions of

$$2^a + 2^b = m^p$$

in which a, b, m, p are non–negative integers, $a \neq b$ and $p \geq 3$.

SOLUTION By symmetry, we may assume that $a < b$. Then

$$2^a + 2^b = 2^a(1 + 2^c)$$

with $c = b - a \geq 1$. Since the first factor is a power of 2 and the second is odd, they have no common factor. So the only way for the product to be a pth power is for each factor to be a pth power. Considering the second factor, this means that

$$1 + 2^c = n^p$$

for some integer n; it is not hard to see that n is odd and $n \geq 3$. Therefore,

$$2^c = n^p - 1 = (n-1)(n^{p-1} + n^{p-2} + \cdots + n + 1)\,.$$

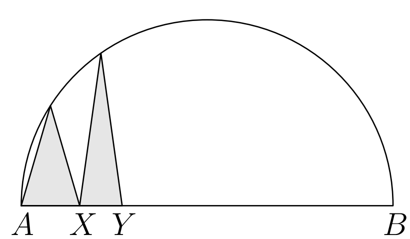
Since the product on the right-hand side equals a power of 2, each factor is a power of 2; neither factor can be $2^0 = 1$, so they are both even. But the second factor is a sum of p odd numbers, and for this to be even, p must be even.

So let $p = 2q$. Then the previous equation can be rewritten as

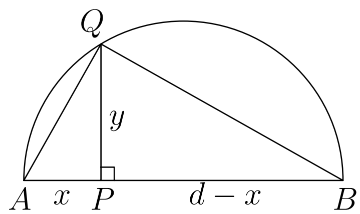
$$2^c = n^{2q} - 1 = (n^q - 1)(n^q + 1)\,.$$

Therefore, $n^q - 1$ and $n^q + 1$ are powers of 2, and their difference is 2: the only possibility is that $n^q - 1 = 2$ and $n^q + 1 = 4$, so $n^q = 3$; therefore, $n = 3$ and $q = 1$. But this means $p = 2$, which is not so. We have ruled out all possibilities, and therefore the required equation has no solution.

Q1693 In a semicircle on diameter AB we have $AX = 3$ and $XY = 2$. As shown in the diagram, two isosceles triangles of equal area have AX and XY as their bases, and their third vertices are on the semicircle. Find the diameter of the semicircle.



SOLUTION Let the diameter of the semicircle be d. In the following diagram, triangles APQ and QPB are similar,



and so

$$\frac{x}{y} = \frac{y}{d-x}\,, \quad \text{that is,} \quad y^2 = x(d-x)\,.$$

So in the diagram from the question, the areas of the two triangles are

$$\frac{1}{2} \times 3\sqrt{\frac{3}{2}\Big(d-\frac{3}{2}\Big)} \quad \text{and} \quad \frac{1}{2} \times 2\sqrt{4(d-4)}\,.$$

Since these areas are equal we have

$$9\Big(\frac{3}{2}\Big)\Big(d-\frac{3}{2}\Big) = 4(4)(d-4)\,,$$

which simplifies to

$$27(2d-3) = 64(d-4)$$

and gives the diameter $d = \dfrac{35}{2}$.

Alternative solution, submitted by Hyunbin Yoo, South Korea. Let C and D be the upper vertices of the triangles on bases AX and XY respectively, and let M and N be the midpoints of these bases. Since triangles ACX and XDY have equal areas and their bases are in proportion $3:2$, their altitudes must be in proportion $2:3$. So we can write $CM = 2h$ and $DN = 3h$. If the radius of the circle is r and the centre O, then Pythagoras' Theorem in the right–angled triangles CMO and DNO gives

$$(2h)^2 + \Big(r-\frac{3}{2}\Big)^2 = r^2 \quad \text{and} \quad (3h)^2 + (r-4)^2 = r^2\,;$$

eliminating h and solving gives diameter $2r = \dfrac{35}{2}$ as above.

Q1694 Let a, m and n be positive integers, where m is odd. Find the greatest common factor of $a^m - 1$ and $a^n + 1$.

SOLUTION Let g be the required greatest common factor: then we have

$$a^m - 1 = sg \quad \text{and} \quad a^n + 1 = tg$$

for some integers s, t. Rewriting the first equation and using the binomial theorem, we have

$$\begin{aligned} a^{mn} &= (1 + sg)^n \\ &= 1 + \binom{n}{1} sg + \binom{n}{2} s^2 g^2 + \cdots + s^n g^n \\ &= 1 + g\left[\binom{n}{1} s + \binom{n}{2} s^2 g + \cdots + s^n g^{n-1}\right]. \end{aligned}$$

Since the expression in square brackets is an integer, g is a factor of $a^{mn} - 1$. By a similar process, and remembering that m is odd, we have

$$a^{mn} = (-1 + tg)^m = -1 + g\left[\binom{m}{1} t - \binom{m}{2} t^2 g + \cdots + t^m g^{m-1}\right],$$

and so g is a factor of $a^{mn} + 1$. Therefore g is a factor of the difference

$$\bigl(a^{mn} + 1\bigr) - \bigl(a^{mn} - 1\bigr) = 2 ,$$

and so $g = 1$ or $g = 2$. Now if a is odd, then $a^m - 1$ and $a^n + 1$ are even, so $g = 2$ is a factor of each; if a is even, then these numbers are odd, so $g = 2$ is a factor of neither. Therefore, for any odd m, the greatest common factor of $a^m - 1$ and $a^n + 1$ is 1 if a is even, 2 if a is odd.

Q1695 Three groups of 2, 3 and 4 passengers arrive independently and at random times at a railway station where a train departs every 12 minutes. What is the probability that the average waiting time per person is more than 8 minutes?

SOLUTION Suppose that the three groups arrive at x minutes, y minutes and z minutes before the next train is due. The average waiting time will then be

$$\frac{2x + 3y + 4z}{9} ,$$

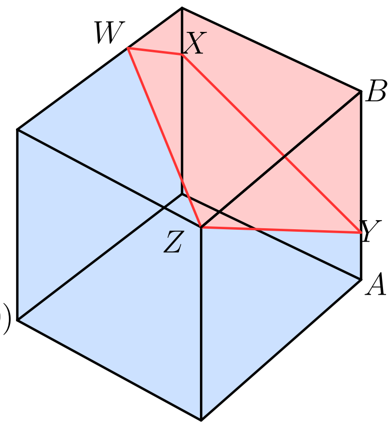
and we want to find the probability that $2x + 3y + 4z > 72$. The space of all possible arrival times for the three groups can be visualised as the cube specified by the inequalities

$$0 \le x \le 12 , \quad 0 \le y \le 12 , \quad 0 \le z \le 12 ,$$

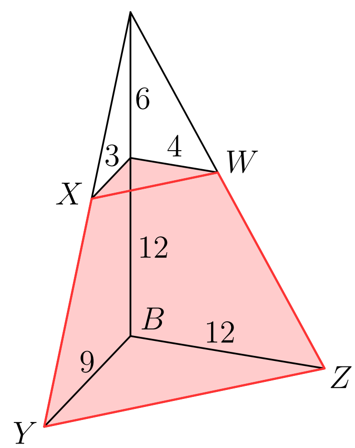
and the probability we want is the proportion of this cube which satisfies the inequality $2x + 3y + 4z > 72$. Geometrically, this is the region within the cube and above the plane

$2x+3y+4z = 72$. The plane intersects the edges of the cube at the four points W, X, Y, Z in the diagram.

$$W = (0, 8, 12)$$
$$X = (0, 12, 9)$$
$$Y = (12, 12, 3)$$
$$Z = (12, 0, 12)$$



For example, to find the point Y, we note that it lies on the edge AB, where we have $x = 12$, $y = 12$: substituting into the equation of the plane gives $z = 3$. Calculations for W, X and Z are left to the reader. The region that we are seeking is a truncated pyramid; the base is a right–angled triangle and the vertex is perpendicularly above the right angle. Constructing a similar pyramid on top of the truncated pyramid,



we find the volume to be

$$\frac{12 \times 9 \times 18}{6} - \frac{4 \times 3 \times 6}{6} = 26 \times 12\,.$$

The required probability is the ratio of this volume to the whole cubical volume,

$$p = \frac{26 \times 12}{12^3} = \frac{13}{72}\,.$$

Q1696 Let n be an integer, $n \geq 2$, and let $p(x)$ be a polynomial with degree at most n, having integer coefficients. Suppose that the values of $p(x)$, where x is an integer, include all the numbers $0, 1, 2, \ldots, n$. Prove that $p(x) = x + c$ for some constant c.

SOLUTION Begin with the well-known factorisation

$$x^m - y^m = (x - y)(x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}) .$$

This shows that if x and y are integers then $x - y$ is a factor of $x^m - y^m$; multiplying by integer coefficients and adding up a number of terms like this shows that $x - y$ is always a factor of $p(x) - p(y)$.

Now the values of $p(x)$ include all the numbers $0, 1, 2, \ldots, n$, say

$$p(a_0) = 0 , \quad p(a_1) = 1 , \quad p(a_2) = 2 , \ldots , \quad p(a_n) = n .$$

All the values a_k must be different since all the values $p(a_k)$ are different. Now from the previous paragraph, $a_1 - a_0$ is a factor of $p(a_1) - p(a_0)$; that is, $a_1 - a_0$ is a factor of 1; and since a_1, a_0 are integers, $a_1 - a_0 = \pm 1$. Treating other pairs in the same way, we have

$$a_1 - a_0 = \pm 1 , \ a_2 - a_1 = \pm 1 , \ldots , \ a_n - a_{n-1} = \pm 1 .$$

Moreover, all the signs in these equations must be the same, or else we should have at some point $a_{k+1} - a_k = -(a_{k+2} - a_{k+1})$, so $a_k = a_{k+2}$, which is not true. We shall treat the case in which all the signs in the equations are $+$; the case in which they are all $-$ is very similar.

So then we have $a_k = a_0 + k$ for all k, and hence

$$p(a_k) - a_k + a_0 = 0$$

for all k. This means that the polynomial $p(x) - x + a_0$ has a minimum of $n + 1$ roots. But this polynomial has degree n or less, and so the only possibility is that it is zero for all values of x. Therefore, $p(x) = x - a_0$, and this is x plus a constant, as required.

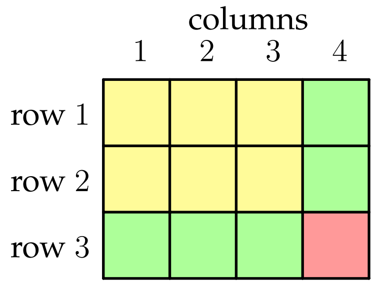
Q1697

(a) Each cell in a 3×4 rectangle has a value of 0 or 1. Prove that it is impossible for every row and column to have an odd sum. How many such rectangles can be constructed such that each row and column has an even sum?

(b) Each cell in $3 \times 4 \times 5$ box is assigned an integer value from 0 to 9. How many such boxes can be formed so that every line in all three directions has a sum divisible by 10?

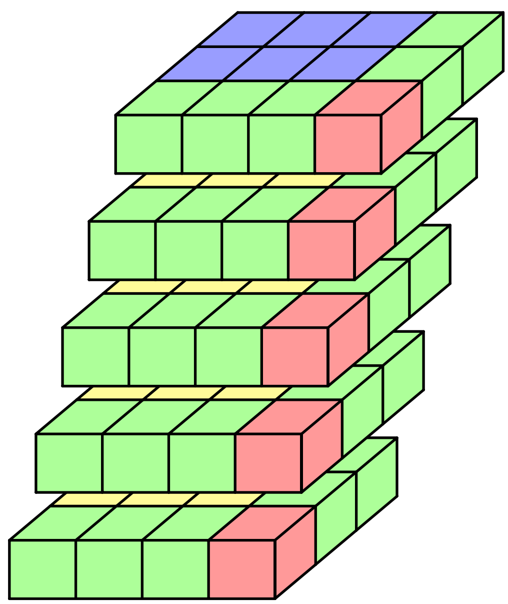
SOLUTION Let S be the sum of all cells, and suppose that all row and column sums are odd. Then S is both the sum of three odd numbers and the sum of four odd numbers; so S is simultaneously odd and even. This is clearly impossible, and we have answered the first question.

Now we count the rectangles in which the sum of each row and column is even.



There are clearly $2^6 = 64$ ways to enter either 0 or 1 in each of the yellow cells. We can then fill in uniquely the bottom entries in columns 1, 2 and 3 to make sure that each of these columns has an even sum; and then the entries in column 4 to make sure that every row has an even sum. We need to check that column 4 also has an even sum. But we can obtain the sum of column 4 by adding all the entries in the grid, which have an even total; and then subtracting the entries in columns 1, 2, 3, which also have an even total; so the sum of column 4 is indeed even. Thus, every choice of six numbers for the yellow cells gives exactly one completed grid, and so the number of ways to fill the grid is 64.

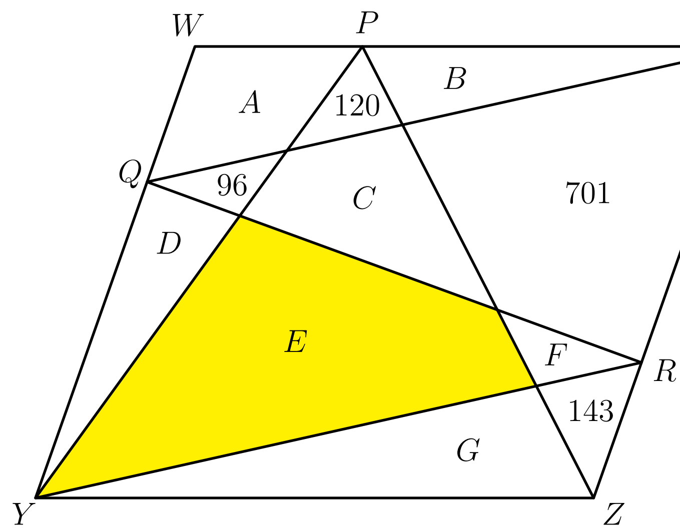
For part (b), we consider five "layers", each of which is a 3 by 4 rectangle as in part (a).



For each of the $2 \times 3 \times 4$ yellow cells in the bottom four layers, enter a number from 0 to 9. There are 10^{24} ways of doing this, and by an argument similar to that in (a), each of these gives one way to fill the remaining cells in these layers so that every horizontal line in these layers adds up to a multiple of 10. The numbers in the yellow cells also give exactly one possibility for the blue cells above them in the top layer, and these then give one way to fill the green and red cells in the top layer so that all horizontal rows in the top layer add up to a multiple of 10.

It remains to check that for every vertical line of green or red cells, the sum is also a multiple of 10. Essentially, this is the same argument as we have used previously. For example, consider the back face of the box (hidden in the diagram). This consists of 5 horizontal lines of four cells: we know that each line sums to a multiple of 10, so the whole face sums to a multiple of 10. But the face also consists of 4 vertical lines of five cells: we know that 3 of them (the ones with a blue cell on top) sum to a multiple of 10, and therefore so does the last. The number of ways to fill the grid is 10^{24}.

Q1698 In the diagram, $WXYZ$ is a parallelogram, and the numbers indicate the areas of certain subregions. Find the area of the coloured region.



SOLUTION submitted by Hyunbin Yoo, South Korea. Label the unspecified regions with letters as shown. Let h be the altitude of the parallelogram, that is, the perpendicular distance between WX and YZ. Then

$$\text{area}(PWY) + \text{area}(PXZ) = \frac{(PW)h}{2} + \frac{(PX)h}{2} = \frac{(YZ)h}{2} = \text{area}(YZP)$$

and so area(PWY) + area(PXZ) is half the area of the parallelogram. For similar reasons, area(QWX) + area(QYR) is also half the area of the parallelogram. Substituting the areas marked on the diagram,

$$A + 120 + B + D + E + F = A + 96 + D + B + 701 + F + 143 \ ,$$

which simplifies to $E = 820$.

Q1699 Suppose that

$$\left(\sqrt{20}+\sqrt{23}\right)^{2023} = a\sqrt{20}+b\sqrt{23}$$

where a and b are integers. Find the remainders when a and b are divided by 33.

SOLUTION Note that

$$\begin{aligned} a_{k+1}\sqrt{20}+b_{k+1}\sqrt{23} &= \left(\sqrt{20}+\sqrt{23}\right)^{2k+3} \\ &= \left(\sqrt{20}+\sqrt{23}\right)^{2k+1}\left(\sqrt{20}+\sqrt{23}\right)^{2} \\ &= \left(a_k\sqrt{20}+b_k\sqrt{23}\right)\left(43+2\sqrt{20}\sqrt{23}\right) \\ &= (43a_k+46b_k)\sqrt{20}+(40a_k+43b_k)\sqrt{23}\ , \end{aligned}$$

and so

$$a_{k+1} = 43a_k+46b_k\ , \quad b_{k+1} = 40a_k+43b_k\ .$$

Using these recurrences to calculate successive values of (a_k, b_k), and replacing the values by their remainders when divided by 33, we find

$$\begin{aligned} (a_0,b_0) &= (1,1) \\ (a_1,b_1) &= (89,83) \equiv (23,17) \\ (a_2,b_2) &\equiv (1771,1651) \equiv (22,1) \\ &\ \vdots \\ (a_{10},b_{10}) &\equiv (1,1)\ . \end{aligned}$$

Thus the remainders of a and b when divided by 33 repeat every 10 steps. To find the values of a and b in the question, we take $2k+1=2023$, that is, $k=1011$, and we have

$$(a,b) = (a_{1011},b_{1011}) \equiv (a_1,b_1) = (23,17)\ .$$

That is, a has remainder 23 when divided by 33, and b has remainder 17.

Q1700 Find the sum of all natural numbers from 1 to 100 which have no common factor with 2022. Also, write the product of these numbers as an expression in terms of powers and factorials.

SOLUTION We can factorise $2022 = 2\times 3\times 337$, so the numbers we are considering will be the odd numbers from 1 to 100 not divisible by 3. So the sum is given by

$$\begin{aligned} S &= (1+3+5+\cdots+99)-(3+9+15+\cdots+99) \\ &= \frac{50\times 100}{2}-\frac{17\times 102}{2} \\ &= 1633 \end{aligned}$$

and the product by

$$
\begin{aligned}
P &= \frac{1 \times 3 \times 5 \times \cdots \times 99}{3 \times 9 \times 15 \times \cdots \times 99} \\
&= \frac{1}{3^{17}} \frac{1 \times 3 \times 5 \times \cdots \times 99}{1 \times 3 \times 5 \times \cdots \times 33} \\
&= \frac{1}{3^{17}} \frac{1 \times 2 \times 3 \times \cdots \times 100}{1 \times 2 \times 3 \times \cdots \times 34} \frac{2 \times 4 \times 6 \times \cdots \times 34}{2 \times 4 \times 6 \times \cdots \times 100} \\
&= \frac{1}{3^{17}} \frac{1 \times 2 \times 3 \times \cdots \times 100}{1 \times 2 \times 3 \times \cdots \times 34} \frac{2^{17}}{2^{50}} \frac{1 \times 2 \times 3 \times \cdots \times 17}{1 \times 2 \times 3 \times \cdots \times 50} \\
&= \frac{1}{2^{33} 3^{17}} \frac{100!\, 17!}{34!\, 50!} \,.
\end{aligned}
$$

Solutions were received from Hyunbin Yoo, South Korea, and from Shivam Mokashi, Abhinava Vidyalaya, India.