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# A short derivation of the quadratic formula

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## 1 Introduction

Lewis Carroll used to lie in bed at night turning mathematical problems over in his mind [1]. About fifteen years ago, as a way to handle my lifelong insomnia, I began doing the same thing. My eyes-closed research program usually aims at finding alternative proofs of old theorems, such as the identity $\sin^2\theta + \cos^2\theta = 1$, my curious proof of which is found in [2] (for discussions of this proof, see [3] and [4]).

Recently, I fell asleep having derived the quadratic formula in a way that I thought elegant enough to share with *Parabola* readers. The derivation follows quite a different logic than the usual derivation via completing the square. Below, I will present this derivation, contrast it with completing the square, and offer some pedagogical observations about the method.

## 2 A short derivation of the quadratic formula

**Step 1.** The following identity may be verified by expanding and simplifying:

$$(2ax+b)^2 - 4a(ax^2+bx+c) = b^2 - 4ac\,. \tag{1}$$

I leave this calculation to the reader. Note that $a$, $b$, $c$ and $x$ may be real or complex.
**Step 2.** The above identity holds for arbitrary values of the variables. But suppose in particular that $x$ satisfies $ax^2+bx+c=0$. Then (1) implies

$$\begin{aligned} & (2ax+b)^2 = b^2-4ac \\ \Rightarrow\quad & 2ax+b = \pm\sqrt{b^2-4ac} \\ \Rightarrow\quad & 2ax = -b \pm \sqrt{b^2-4ac} \\ \Rightarrow\quad & x = \tfrac{-b\pm\sqrt{b^2-4ac}}{2a}\,. \end{aligned}$$

This is the quadratic formula.

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## 3 Comparison to completing the square

The method of completing the square runs as follows, in the description from [5]:

> The difficulty with the general quadratic equation ($ax^2 + bx + c = 0$ as we write it today) is that, unlike a linear equation, it cannot be solved by arithmetic manipulation of the terms themselves: a creative intervention, the addition and subtraction of a new term, is required to change the form of the equation into one which is arithmetically solvable. We call this maneuver *completing the square*.
>
> Here is how students learn about the maneuver today:
>
> The coefficient $a$ is non-zero (or else the equation is linear) so you can divide through by $a$, giving $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Now comes the maneuver: you add and subtract $\frac{b^2}{4a^2}$, giving $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$. Now the first three terms are a perfect square: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = (x + \frac{b}{2a})^2$: you have completed the first two terms to a square. Instead of having both the unknown $x$ and its square $x^2$ in the equation, you have only the second power $(x+\frac{b}{2a})^2$ of a new unknown. Now arithmetic can do the rest. The equation gets rearranged as $(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a}$ so $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$ giving the solution $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

By contrast, the method that starts with identity (1) isn't a process of transforming the equation $ax^2 + bx + c = 0$ into a form that can be solved using arithmetical steps. Instead, we begin with an identity that is true for every value of $x$, at which point specializing the identity to the case where $x$ satisfies $ax^2 + bx + c = 0$ yields an equation with the same critical feature as in the strategy of completing the square, namely, only the second power of an unknown remains.

## 4 Where did the identity come from?

The origin of identity (1) is immaterial to the logic of the derivation, but curiosity is only natural, so I'll say a word about that. I devised identity (1) by imagining $x$ in the expression $ax^2 + bx + c$ as the time variable for the motion of a projectile. (See [6] for a summary of the equations of projectile motion.) Conservation of energy for a projectile says that the sum of the projectile's kinetic energy and potential energy is constant in time. The kinetic energy of a projectile is quadratic in its velocity (which depends linearly on time), while the potential energy is linear in its height (which depends quadratically on time). Thus, the term $(2ax + b)^2$ in identity (1) is analogous to kinetic energy, while the term $-4a(ax^2 + bx + c)$ is analogous to gravitational potential energy. That makes the quantity $b^2 - 4ac$ analogous to the constant value of total energy. These considerations could be elaborated for a physics-focused audience, but I'll forego that discussion as my interest here has been mathematics.[2]

[2] I looked for an identity analogous to (1) involving third or fourth powers of $x$ that would allow for a simple derivation of the cubic or quartic formulas, but I neither found such an identity nor proved that

## 5 Pedagogical considerations

I am presenting the derivation based on (1) because I find it pleasing as a piece of mathematics, not because I believe it would be pedagogically useful. However, since the quadratic formula is a fixture of school mathematics, it is worth commenting on the pedagogical implications, if any, of an alternative derivation of that formula.

The logic of completing the square is a process of transforming a difficult equation into an easier one, which is a much more straightforward process than the logic of the derivation based on (1). On the other hand, the derivation based on completing the square is much more demanding in terms of procedural fluency: consider the profusion of fractions in that approach, and the necessity of doing manipulations under the square root sign, as in $\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \to \sqrt{\frac{b^2-4ac}{4a^2}}$.

The derivation based on (1) may be unsatisfying, to the extent that the animating identity comes out of nowhere. On the other hand, the technique of completing the square may also appear to come out of nowhere, the first time one sees it.

I suspect that students would need extensive support to understand the idea of an identity before diving into the manipulations of the derivation based on (1). I also suspect that students would need support to grasp the logic of the transition from Step 1 to Step 2 of the derivation. Even with such support, an activity based on this derivation might not be attractive to many students. In the Appendix, I have sketched such an activity—not as a model of a classroom-ready resource, but rather as way to flesh out what I mean when I refer to necessary supports of the ideas.

The derivation based on (1) wouldn't make a good replacement for completing the square in the high school algebra curriculum. In addition to the foregoing reasons, this is because completing the square is historically important [5], and because the technique becomes even more useful in function contexts, where it can be used to rewrite quadratic functions, for example to reveal their extreme values or to locate the vertex of the parabolic graph of the function [7].

Whatever the pedagogical utility of this derivation, on the night it came to me I felt it was worth getting out of bed for. I hope the reader has found it worthwhile too, as a fresh look at an old problem.

## Acknowledgements

I gratefully thank my family members for trying out an earlier version of the notional worksheet in the Appendix; their input was important for improving the earlier version. I'm also grateful to several friends who read an earlier draft of this article, and whose positive reactions gave me the confidence to try writing for *Parabola*.

one cannot exist.

## References


[1] L. Carroll, *Mathematical Recreations of Lewis Carroll: Pillow Problems and A Tangled Tale*, Dover Publications, New York, 1958.

[2] J. Zimba, On the possibility of trigonometric proofs of the Pythagorean Theorem, *Forum Geometricorum* **9** (2009), https://forumgeom.fau.edu/FG2009volume9/FG200925.pdf, last accessed on 2023-08-02.

[3] M. Raman-Sundström and L.-D. Öhman, Mathematical fit: A case study, *Philosophia Mathematica* **26**(2) (2018), 184–210.

[4] C.Z. Elgin, *True Enough*, MIT Press, Cambridge, 2017, p. 202.

[5] T. Phillips, Completing the square: the prehistory of the quadratic formula, *American Mathematical Society Feature Column* (November 1 2020), http://www.ams.org/publicoutreach/feature-column/fc-2020-11, last accessed on 2023-08-02.

[6] P. Bos, Projectiles, *Parabola* **13**(2) (1977), https://www.parabola.unsw.edu.au/1970-1979/volume-13-1977/issue-2/article/projectiles, last accessed on 2023-08-02.

[7] Student Achievement Partners, *Coherence Map*, entry for Common Core State Standard HS.F-IF.C.8.a, https://achievethecore.org/coherence-map/HS/F/109/555/555, last accessed on 2023-08-02.

# Project: Another way to derive the quadratic formula

You may have seen a derivation of the quadratic formula that uses the method of completing the square. In this worksheet, you will derive the formula in a way that doesn't need as many steps. This method starts with an *algebraic identity*. So we should first learn what an algebraic identity is.

## Algebraic identities

*Definition.* An identity is a special kind of equation that is true no matter what the values of the variables are. Here is an example of an identity:

$$x = x$$

This equation is true no matter what the value of $x$ is. That makes it an identity.

*Solving vs. verifying*. Unlike ordinary equations, identities aren't asking to be solved. You don't have to find values that make an identity true, because an identity is true no matter what the values are! But sometimes we have to verify identities, to make sure that they are really true no matter what the values of the variables are.

*How to verify an identity*. The usual way to verify an identity is to simplify and rewrite the expressions on one or both sides, until the same expression is on both sides of the equal sign. Let's try it.

$$x^2 - (x+3)(x-3) = 9.$$

Is this an identity? To verify it, we can expand and simplify the left-hand side and see if the result is the right-hand side.

$$\begin{aligned} x^2 - (x+3)(x-3) &= x^2 - (x^2 - 3x + 3x - 9) \\ &= x^2 - (x^2 - 9) \\ &= x^2 - x^2 + 9 \\ &= 9\,. \end{aligned}$$

We have shown that the identity is true, because we have shown that the left-hand side does equal the right-hand side no matter what the value of $x$ is. Notice that our verification steps didn't assume anything about the numerical value of $x$.

*Disproving an identity*. What if a supposed identity isn't actually true for all values of the variables? One way to find that out is to choose a value for the variable, substitute it into the supposed identity, and find that the resulting equation is false. For example, suppose I tried to tell you that $x^3 = x^2$ is an identity. You could disprove that by saying, "Oh yeah, what if $x$ is 2? Then the equation says $8 = 4$, which is false." Your observation disproves the identity, because an identity has to be true no matter what the values of the variables are. An identity that is only true for special values of the variables is just an ordinary equation.

## A useful identity for deriving the quadratic formula

Here is an identity with four variables in it:

$$(2ax+b)^2 - 4a(ax^2+bx+c) = b^2 - 4ac\,. \qquad (*)$$

Verify that the identity is true for all values of the variables by expanding and simplifying terms on the left-hand side.

## Deriving the quadratic formula from the identity

Now that you have shown that the identity $(*)$ is true for every possible value of $x$ (plus the other three variables), imagine choosing a *special* value of $x$, a value that solves the quadratic equation $ax^2+bx+c=0$. If you substitute this special value of $x$ into the identity $(*)$, then the equation that results will have to be true. And notice: when you substitute the special value of $x$ into the identity $(*)$, the term $-4a(ax^2+bx+c)$ in the identity will equal zero, because the special value of $x$ solves the quadratic equation $ax^2+bx+c=0$.

Therefore, if $x$ solves the quadratic equation $ax^2+bx+c=0$, then the identity $(*)$ implies for this $x$,

$$(2ax+b)^2 = b^2 - 4ac\,. \qquad (**)$$

Now we are nearing the end of our project. Solve equation $(**)$ for $x$, and you should obtain a familiar result.