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Unlocking a new realm of Pythagorean triples

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Welcome, fellow math enthusiasts! Today, I am excited to share with you a disruptive discovery in the world of geometry. By using sequences of fractions arising from periodic continued fractions as seeds to generate Pythagorean triples, I have unlocked a new realm of Pythagorean triples.

As you know, a Pythagorean triple[2] is a set of three integers $[a, b, c]$ that describe the three *integer* side lengths of a right-angled triangle, such as

$$[3, 4, 5], \quad [20, 21, 29], \quad [39, 80, 89].$$

All of these triples satisfy the condition $a^2 + b^2 = c^2$ by the Pythagorean Theorem. The Pythagorean triples seem pretty random at first sight, though there are to date three known families of Pythagorean triples where each triple $[a, b, c]$ satisfies some defining condition:

The *Pythagoras family* satisfies $c - a = 1$.

The *Plato family*[3] satisfies $c - a = 2$.

The *Fermat family*[4] satisfies $|b - a| = 1$.

First, I will present the current methods for generating Pythagorean triples and introduce a new method using sequences of fractions and simple lattice diagrams. Since periodic continued fractions naturally generate sequences of fractions, with a selection of iconic examples I will show you how they naturally generate new exciting families of Pythagorean triples. In the process, I will demonstrate that the famous *silver ratio* generates all the triples of the Fermat family. Ultimately, I will propose a very simple and visual Pythagorean tree whose branches neatly illustrate the multiplicity of Pythagorean families. So, get ready to dive into the fascinating world of Pythagorean triples and join me on this exciting journey of discovery!

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[1] Frédéric Beatrix runs his own architectural firm *blue.archi* in Villefranche-Sur-Mer, France.

[2] Actually, the knowledge of Pythagorean triples predates Pythagoras by more than a thousand years. The oldest known record comes from Plimpton 322, a Babylonian clay tablet from 1800 BC. [11]

[3] The Plato and Pythagoras families are named after the identifications given by Proclus (412-485), in his commentary to the 47th Proposition of the first book of Euclid's Elements

[4] Pierre de Fermat (1607-1665) was a French mathematician who is given credit for early developments that led to infinitesimal calculus, including his technique of adequality.

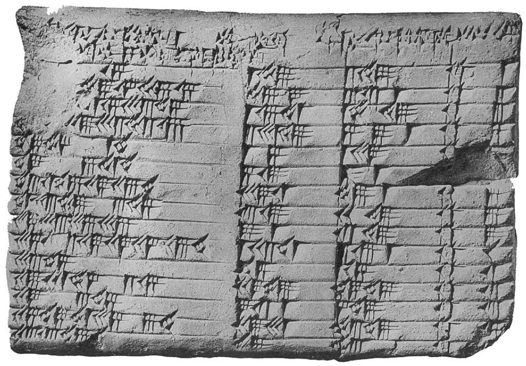

Figure 1: *Babylonian clay tablet showing Pythagorean triples. 1800 BCE*

1 Definitions

Simple continued fraction

A *simple continued fraction* has the following infinite form, where all a_n are integers:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

I use the notation $[a_0; a_1, a_2, a_3, \ldots]$ to represent this continued fraction.

Periodic continued fraction

The numbers with a *periodic continued fraction expansion* are precisely the irrational solutions of quadratic equations with rational coefficients. This is known as *Lagrange's Continued Fraction Theorem*, which was first proved by Joseph-Louis Lagrange (1736-1813) in 1770. The theorem states that each positive quadratic surd $\sqrt{a}$ has a simple continued fraction that is periodic after some point.

I use a bar to indicate the periodically repeating numbers; for instance, $[1; 2, \overline{3, 4}]$ represents the periodic continued fraction $[1; 2, 3, 4, 3, 4, \ldots]$.

Primitive Pythagorean triple

A *primitive Pythagorean triple* is one in which a, b, and c are coprime – that is, they have no common divisor larger than one – and $a + b$ is odd. For example, $[3, 4, 5]$ is a primitive Pythagorean triple whereas $[6, 8, 10]$ is not.

Peacock diagram

An *arithmetical lattice diagram* is a geometrical assembly of trigons where two initial values a and b are assigned to the first two vertices. The number assigned to the third vertex is $a+b$. Then the value of each consecutive vertex on the adjacent trigon is the sum of the numbers given to the previous two vertices.

I call a *peacock diagram* an arithmetical lattice diagram designed to visualise periodic continued fractions. Let's consider $\sqrt{x} = [\frac{a_n}{2}; \overline{a_1, a_2, \ldots, a_n}]$. I will use a motif composed of n peacock-tail-shapes, alternatively upward and downward, where each individual number $\overline{a_k}$ of the period is represented by *one* "peacock-tail" split into a_k equal parts. The repetition of the motif creates an infinite chain.

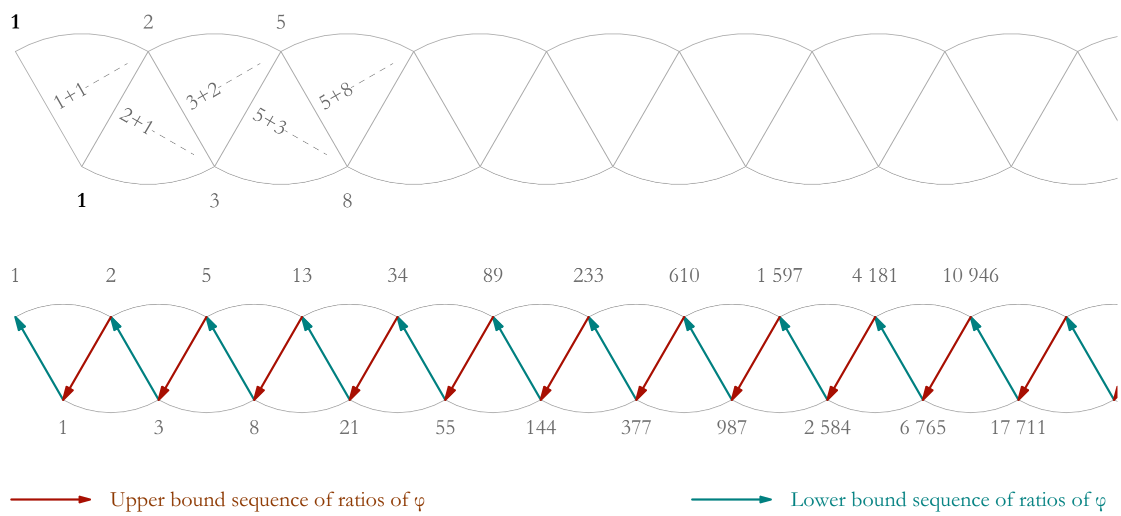

Figure 2: *Peacock diagram showing converging ratios approximating* φ

The arithmetic lattice diagram of the golden ratio $\varphi = [1; \overline{1}] = \frac{1+\sqrt{5}}{2}$ illustrated in Figure 2 adequately illustrates the generic structure of the Peacock diagram.

Since we can choose any segment to assign the initial value $(1, 1)$ to its two vertices, there are $a_1 + \cdots + a_n$ unique Peacock diagrams in the family of $\sqrt{x}$. An interesting collateral result of this representation is that, for each $\overline{a_k} > 1$, it reveals $a_k - 1$ new sequences of convergent fractions towards $\sqrt{x}$ that were mostly ignored so far[5].

Here are a few examples:

$$\frac{11}{5}, \frac{47}{21}, \frac{199}{89}, \frac{843}{377}, \ldots \text{ converge towards } \sqrt{5}$$
$$\frac{7}{3}, \frac{29}{13}, \frac{123}{55}, \frac{521}{233}, \ldots \text{ converge towards } \sqrt{5}$$
$$\frac{5}{2}, \frac{20}{9}, \frac{85}{38}, \frac{360}{161}, \ldots \text{ converge towards } \sqrt{5}$$

[5]That means plenty to feed into the On-Line Encyclopedia of Integer Sequences (oeis.org).

2 How to generate Pythagorean triples?

Berggren [5] showed in 1934 that any primitive Pythagorean triple can be obtained from the initial triple $[3, 4, 5]$. In 2007, Price and Bernhart [7] demonstrated that fractions of coprime numbers are natural generators of Pythagorean triples. Then in the 2018 paper [6], they also showed that all Pythagorean triples can be calculated from a tree of *Fibonacci boxes*, which are a set of four numbers, all arising from the initial set $[1, 1, 2, 3]$.

Since in this paper I will argue that families of Pythagorean Triples are the fruits of sequences of fractions, I propose the following method based on Euclid's Formula. We name the numerator n and its denominator d, both integers with $n > d$, and we generate the Pythagorean triple as follows:

$$\frac{n}{d} \rightarrow [a, b, c] = [n^2 - d^2\,,\ 2nd\,,\ n^2 + d^2]\,.$$

The first fraction of integers where $n > d$ is necessarily $\frac{2}{1}$. It generates the initial triple:

$$\frac{2}{1} \rightarrow\ [3, 4, 5]\,.$$

3 Algebraic families

Here is a short selection of families of Pythagorean triples arising from sequences of fractions of the type

$$U_n = \frac{an + b}{cn + d}$$

where a, b and c are integers.

3.1 Pythagoras

The *Pythagoras family* arises from the sequence of fractions

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n+1}{n}, \ldots$$

where

$$\frac{n+1}{n} \rightarrow\ [2n + 1\,,\ 2n^2 + 2n\,,\ 2n^2 + 2n + 1]\,.$$

The triples $[a, b, c]$ of this family satisfy $c - a = 1$. Examples include

$$[5, 12, 13]\,,\quad [7, 24, 25]\,,\quad [9, 40, 41]\,.$$

3.2 Euclid

The *Euclid family* arises from the sequence of fractions

$$\frac{3}{1}\,,\ \frac{5}{3}\,,\ \frac{7}{5}\,,\ldots,\ \frac{2n+1}{2n-1}\,,\ldots$$

where

$$\frac{2n+1}{2n-1} \to\ [8n\,,\ 8n^2-2\,,\ 8n^2+2]\,.$$

The triples $[a, b, c]$ of this family satisfy $c - b = 4$. Examples include

$$[16, 30, 34]\,,\quad [24, 70, 74]\,,\quad [32, 126, 130]\,.$$

None of the members of this family are primitive triples.

3.3 Plato

The *Plato family* arises from the sequence of fractions

$$\frac{2}{1}\,,\ \frac{3}{1}\,,\ \frac{4}{1}\,,\ldots,\ \frac{n}{1}\,,\ldots$$

where

$$\frac{n}{1} \to\ [n^2-1\,,\ 2n\,,\ n^2+1]\,.$$

The triples $[a, b, c]$ of this family satisfy $c - a = 2$, and only the members of the form $\frac{2n}{1}$ generate primitive triples. Examples include

$$[15, 8, 17]\,,\quad [35, 12, 37]\,,\quad [99, 20, 101]\,.$$

3.4 Socrates

The *Socrates family* arises from the sequence of fractions

$$\frac{3}{2}\,,\ \frac{5}{2}\,,\ \frac{7}{2}\,,\ldots,\ \frac{2n+1}{2}\,,\ldots$$

where

$$\frac{2n+1}{2} \to\ [4n^2+4n-3\,,\ 8n+4\,,\ 4n^2+4n+5]\,.$$

The triples $[a, b, c]$ of this family satisfy $c - a = 8$. Examples include

$$[21, 20, 29]\,,\quad [45, 28, 53]\,,\quad [77, 36, 85]\,.$$

4 Periodic continued fractions

In the past, mathematicians approximated irrational values such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ by fractions because it was difficult to compute their exact values. For example, the Babylonian clay tablet YBC 7289 (c. 1800–1600 BCE) [1] gave an approximation of $\sqrt{2}$ as the sum of four sexagesimal fractions:

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = \frac{305\,470}{216\,000} = 1.41421296\ldots \approx 1.41421356\ldots = \sqrt{2}\,.$$

This approximation is remarkably accurate. In the third century BCE, Archimedes gave an approximation for $\sqrt{3}$ as $\frac{265}{153} < \sqrt{3} < \frac{1\,351}{780}$ which he used to obtain his approximation for π as $3\frac{10}{71} < \pi < 3\frac{1}{7}$[6]. These approximations were obtained using an algorithm of converging fractions. Similarly, Hero of Alexandria (1st century CE) [3, p. 323] and Isaac Newton (1669) [4] used such an algorithm to approximate irrational values.

All quadratic surds of the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers, have a periodic continued fraction. A simple geometrical algorithm, or a Peacock diagram, will harvest the sequences of converging fractions arising from the said periodic continued fraction. I will show that such sequences of converging fractions will generate new families of Pythagorean triples.

For a geometrical approach, any fraction of integers $\frac{n}{d}$ where $n > d$ can be represented as a rectangle of length n and width d. Therefore, a simple square represents the number 1, and the *Quadratum Lungum*, or double-square, represents the number 2.

We can illustrate the arithmetic of fractions as a sum of rectangles. For example in Figure 3, we can easily add 1 to $\frac{5}{2}$ by adding a square *above* a 5×2 rectangle. Inverting a fraction means swapping the x and y axes on a standard planar graph. Therefore, adding 1 to $\frac{1}{\frac{5}{2}}$ involves adding a square *to the right* of the same rectangle.

4.1 Golden family $\varphi = [1; \overline{1}]$

Should we continue the iterative process of adding squares, as started in Figure 3, alternatively *above* and *to the right*, we would effectively build the continued fraction of the golden ratio φ from the bottom up:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = \frac{1 + \sqrt{5}}{2}\,.$$

The simplicity of the geometrical algorithm, illustrated in Figure 4, is indeed reflected in the more practical notation for this continued fraction: $\varphi = [1; \overline{1}]$. Furthermore, it is visually simple to assess the increasing accuracy of the sequence of ratios approximating φ as shown in green in Figure 4.

[6] Archimedes obtained this approximation in his treatise *Dimension of the Circle* (ca. 250 BCE). The approximation is correct up to 0.001%.

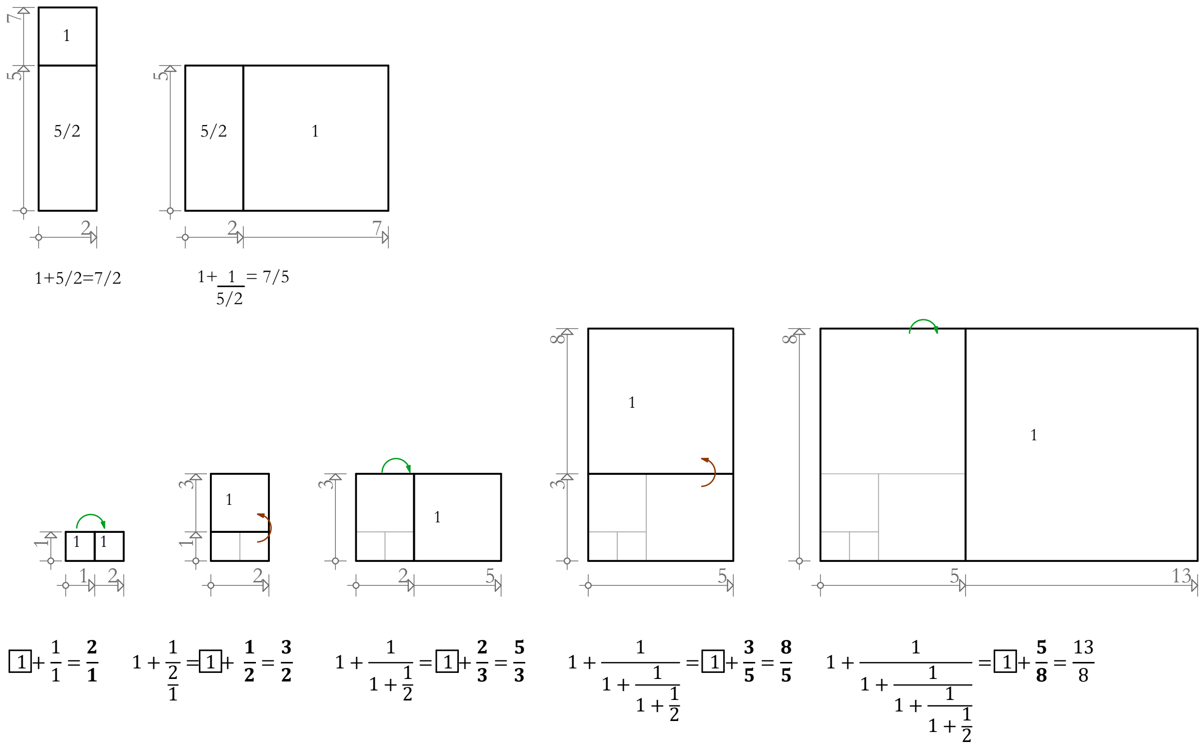

Figure 3: *Basic arithmetic of fractions using geometry*

In this instance, the simple stacking of squares naturally reveals the numbers of the Fibonacci sequence $[1, 1, 2, 3, 5, 8, 13, 21, \ldots]$, or more precisely, two parallel sequences of fractions arising from the consecutive $\frac{\text{length}}{\text{width}}$ fractions of the consecutive rectangles:

$$S_- = \left[\frac{1}{1}, \frac{3}{2}, \frac{8}{5}, \frac{21}{13}, \ldots\right] \tag{1}$$

$$S_+ = \left[\frac{2}{1}, \frac{5}{3}, \frac{13}{8}, \frac{34}{21}, \ldots\right] \tag{2}$$

Interestingly, the vertical quasi-golden rectangles reflected as the sequence S_- always have a proportion which is less than φ. In contrast, the horizontal rectangles reflected in S_+ always have a proportion greater than φ. This periodic stacking of rectangles is similarly reflected in the Peacock diagram in Figure 2.

The Pythagorean triples $[a, b, c]$ generated by the sequences S_- and S_+ are all primitive and they satisfy the condition

$$\frac{|b - 2a|}{2} = 1\,.$$

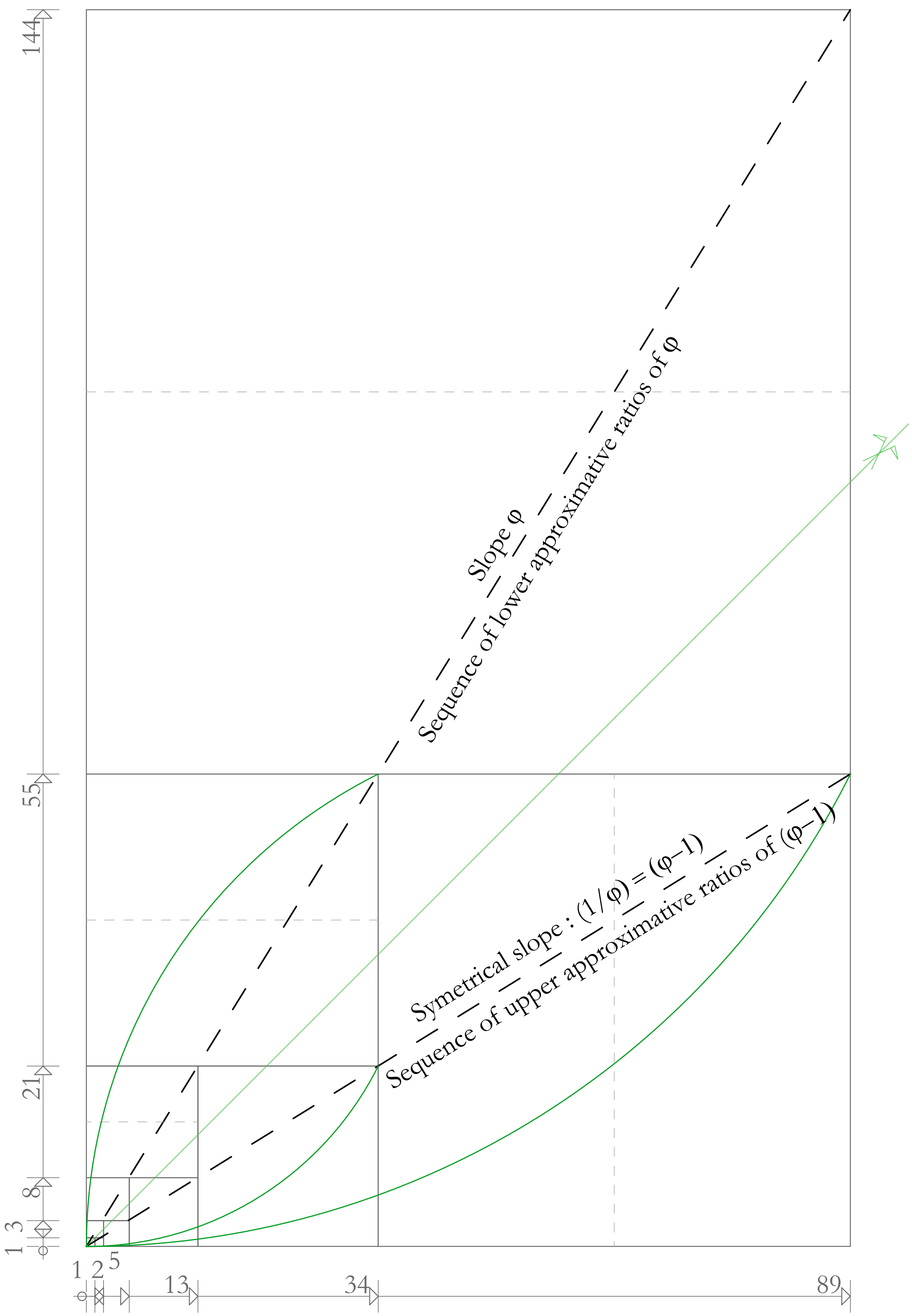

Figure 4: *Stacking squares to get the converging ratios of φ.*

Examples for the lower bound sequence of fractions S_- include

$$\frac{3}{2} \rightarrow [5, 12, 13]$$
$$\frac{8}{5} \rightarrow [39, 80, 89]$$
$$\frac{21}{13} \rightarrow [272, 546, 610]$$

Examples for the upper bound sequence of fractions S_+ include

$$\frac{2}{1} \rightarrow [3, 4, 5]$$
$$\frac{5}{3} \rightarrow [16, 30, 34]$$
$$\frac{13}{8} \rightarrow [105, 208, 233]$$

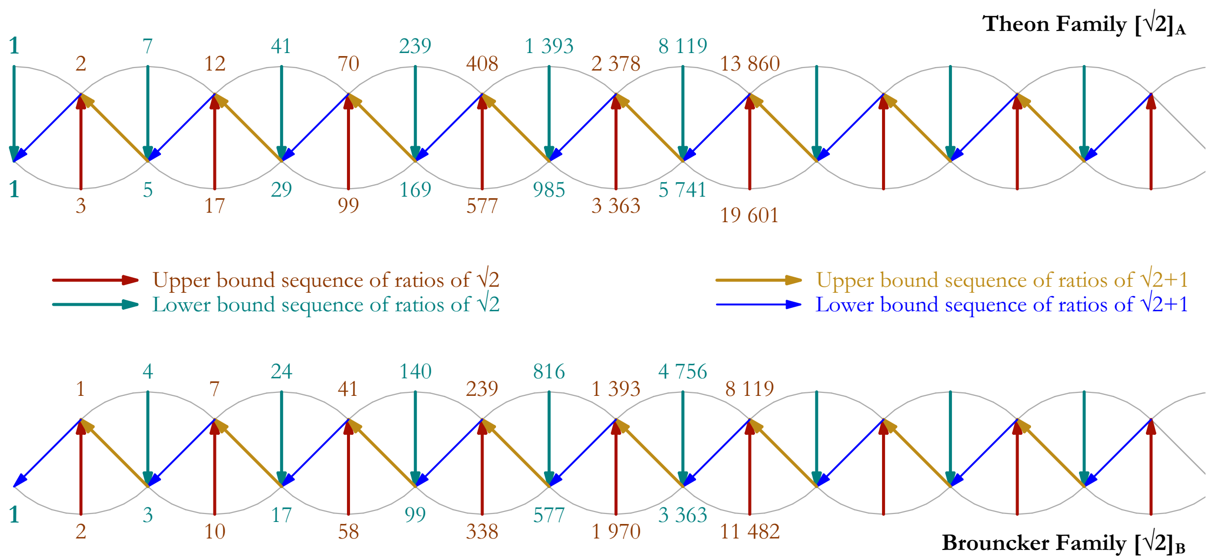

Figure 5: *The two Peacock diagrams for* $[\sqrt{2}]$ *and the Silver Ratio* $[\sqrt{2}+1]$.

4.2 $[\sqrt{2}]$ families of Pythagorean triples

The periodic continued fraction of the silver ratio $\sqrt{2}+1$ has a similar structure to the golden ratio:

$$\sqrt{2}+1 = [2;\overline{2}] = 2 + \cfrac{1}{2+\cfrac{1}{2+\cdots}}\,.$$

We can create two Peacock diagrams where the peacock-tails are divided into two equal parts; see Figure 5. As shown in Figure 13, we could also create an elegant 3D lattice diagram and beautiful nested squares to show the two parallel sequences of converging fractions.

The *Theon family* $[\sqrt{2}]_A$ of Pythagorean triples arises from the sequence of fractions also known as the *Pell Numbers*[7] illustrated in the upper part of Figure 5. We can obtain these numbers with the simple stacking of double-squares as illustrated in Figures 11 and 12. The Theon family satisfies the condition

$$|n^2-2d^2| = \frac{|3a-c|}{2} = 1\,.$$

Examples include

$$\frac{17}{12} \rightarrow [145, 408, 433]$$
$$\frac{41}{29} \rightarrow [840, 2\,378, 2\,522]$$
$$\frac{99}{70} \rightarrow [4\,901, 13\,860, 14\,701]$$

The *Brouncker family*[8] $[\sqrt{2}]_B$ of Pythagorean triples arises from the sequence of fractions illustrated in the lower part of Figure 5. These triples $[a,b,c]$ satisfy the condition

$$|n^2-2d^2| = \frac{|3a-c|}{2} = 2\,.$$

Examples include

$$\frac{24}{17} \rightarrow [287, 816, 865]$$
$$\frac{58}{41} \rightarrow [1\,683, 4\,756, 5\,045]$$
$$\frac{140}{99} \rightarrow [9\,799, 27\,720, 29\,401]$$

[7]The name "Pell numbers" stems from Leonhard Euler's mistaken attribution of the solutions to the equation $nx^2+1=y^2$, and the numbers derived from it when $n=2$, to the English mathematician John Pell (1611-1685). The solutions were already presented at the beginning of the 2nd century CE by Theon of Smyrna in *On Mathematics Useful for the Understanding of Plato*. In Section I, Chapter 31, Theon works out this family of triples from a simple geometric algorithm using the lateral and diagonal dimensions of successive squares [2].

[8]William Brouncker, 2nd Viscount Brouncker FRS (c. 1620 – 5 April 1684) was an Anglo-Irish peer and mathematician who served as the president of the Royal Society from 1662 to 1677.

4.3 Silver families $[2, \overline{2}]$

The *Silver family* $[2, \overline{2}]_A$ that is generated by the sequence of fractions illustrated in the upper part of the same Figure 5 is known today as the *Fermat family* of primitive Pythagorean triples. Its members satisfy the condition

$$|b - a| = 1\,.$$

Examples include

$$\begin{aligned} \frac{29}{12} &\rightarrow [696, 697, 985] \\ \frac{70}{29} &\rightarrow [4\,059, 4\,060, 5\,741] \\ \frac{169}{70} &\rightarrow [23\,660, 23\,661, 33\,461] \end{aligned}$$

The members of the silver family $[2, \overline{2}]_B$ of Pythagorean triples arise from the Brouncker sequence of fractions illustrated in the lower part of Figure 5. These triples $[a, b, c]$ satisfy the condition

$$|b - a| = 2\,.$$

Its non-primitive triples are all doubles of the Fermat Family, such as those below:

$$\begin{aligned} \frac{7}{3} &\rightarrow [40, 42, 58] \\ \frac{17}{7} &\rightarrow [240, 238, 338] \\ \frac{41}{17} &\rightarrow [1\,392, 1\,394, 1\,970] \end{aligned}$$

4.4 Archimedes families [$\sqrt{3}$]

Since the continued fraction of $\sqrt{3}$ is $[1; \overline{1; 2}]$, the periodic motif of its Peacock diagrams is composed of two peacock-tail shapes, one of which is split in two. This generates three unique Peacock diagrams as illustrated in Figure 6. Note the segments $[12, 7, 4]$, $[45, 26, 15]$, $[168, 97, 56]$ and so on showing converging fractions of $\sqrt{3}$ that can generate elegant nested rhombi as illustrated in Figure 19.

Note also that we easily obtain the same lower bound approximation $\frac{265}{153} < \sqrt{3}$ and the upper bound approximation $\sqrt{3} < \frac{1\,351}{780}$ stated by Archimedes. This basically comes down to stacking alternatively squares and double squares, as illustrated in Figures 14-17, where it is visually easy to assess the accuracy of the approximation with a straightforward equilateral triangle.

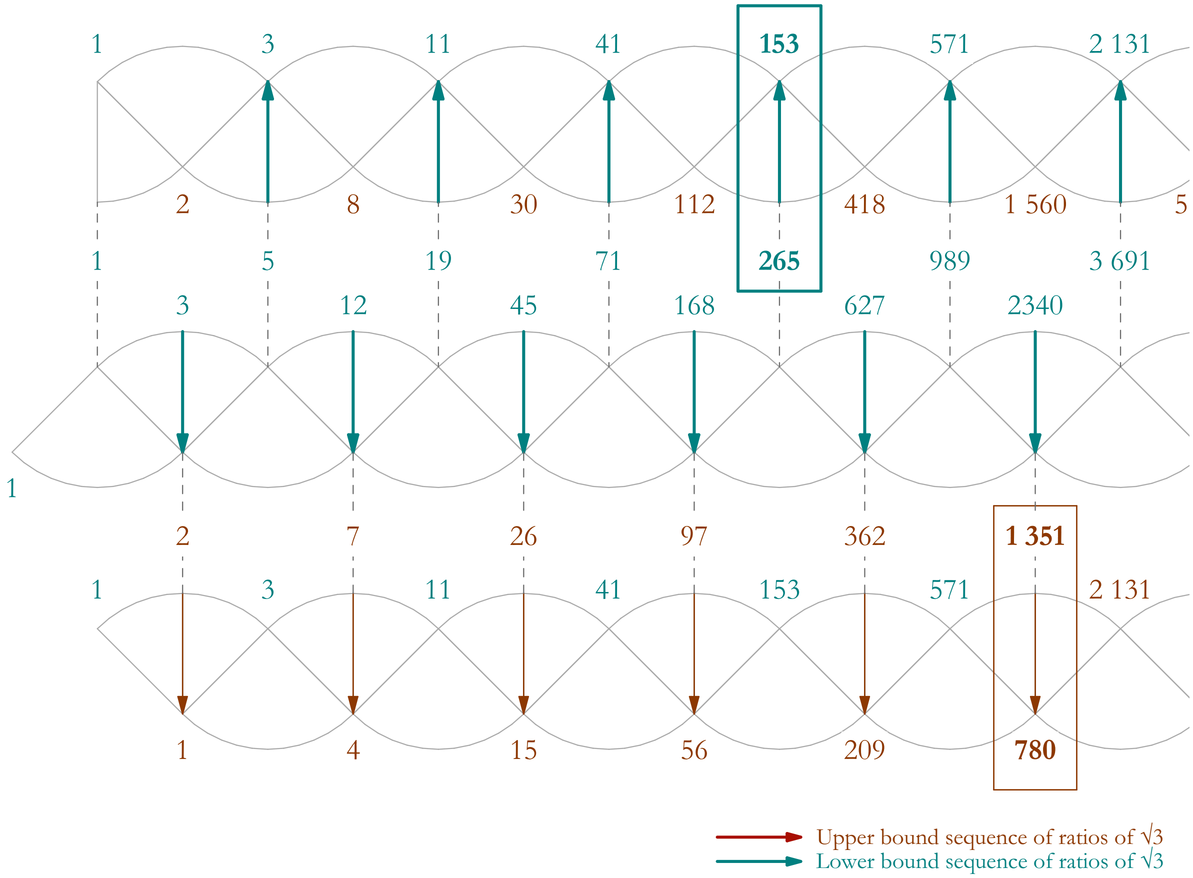

Figure 6: *the three Peacock diagrams showing the converging fractions of* $\sqrt{3}$

The *Archimedes*$_A$ *family* (lower bound sequence) satisfies the condition

$$3d^2 - n^2 = c - 2a = 2\,.$$

Examples include

$$\frac{5}{3} \quad\rightarrow\quad [16, 30, 34]$$
$$\frac{19}{11} \quad\rightarrow\quad [240, 418, 482]$$
$$\frac{71}{41} \quad\rightarrow\quad [3\,360, 5\,822, 6\,722]$$

The *Archimedes*$_B$ *family* (lower bound sequence) satisfies the condition

$$3d^2 - n^2 = c - 2a = 3$$

Examples include

$$\begin{aligned} \frac{3}{2} &\;\rightarrow\; [5, 12, 13] \\ \frac{12}{7} &\;\rightarrow\; [95, 168, 193] \\ \frac{45}{26} &\;\rightarrow\; [1\,349, 2\,340, 2\,701] \end{aligned}$$

The *Archimedes*$_C$ *family* (upper bound sequence) satisfies the condition

$$3d^2 - n^2 = c - 2a = -1\,.$$

Examples include

$$\begin{aligned} \frac{7}{4} &\;\rightarrow\; [33, 56, 65] \\ \frac{26}{15} &\;\rightarrow\; [451, 780, 901] \\ \frac{97}{56} &\;\rightarrow\; [6\,273, 10\,864, 12\,545] \end{aligned}$$

4.5 Architect families [$\sqrt{5}$]

We now consider the continued fraction $\sqrt{5} = [2, \overline{4}]$ which generates four unique Peacock diagrams as illustrated in Figure 7.

The *Theodoros family*[9] $[\sqrt{5}]_A$, satisfies the condition

$$|5d^2 - n^2| = |3a - 2c| = 4\,.$$

Examples include

$$\begin{aligned} \frac{11}{5} &\;\rightarrow\; [96, 110, 146] \\ \frac{47}{21} &\;\rightarrow\; [1\,768, 1\,974, 2\,650] \\ \frac{199}{89} &\;\rightarrow\; [31\,680, 35\,422, 47\,522] \end{aligned}$$

[9] Theodoros was a Greek architect who lived with Pythagoras on the small Aegean island of Samos under the rule of Polycrates, during the 6th century BCE. The floor plan of the Temple of Hera that he built on Samos Island is a *Quadratum Lungum*. [9]

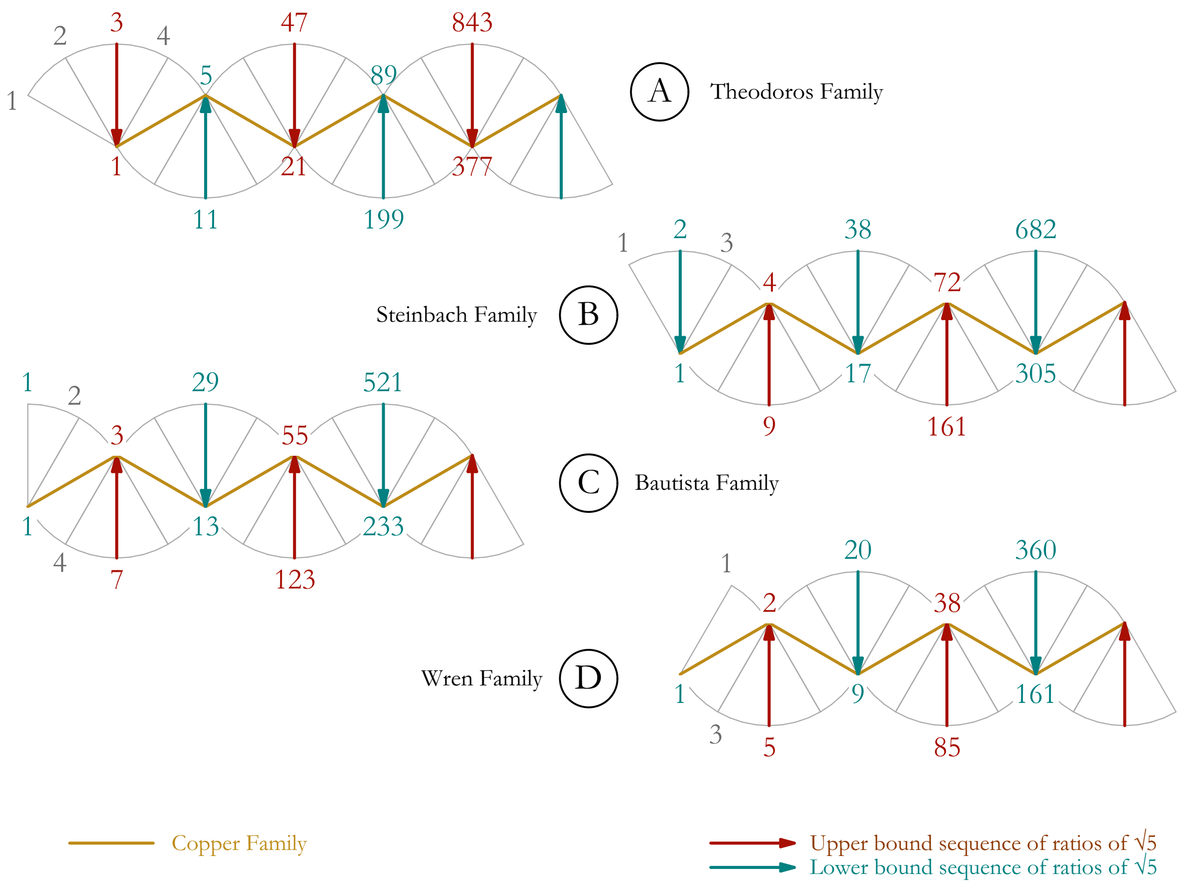

Figure 7: *The four peacock diagrams of converging fractions of* $\sqrt{5}$

The *Steinbach family*[10] $[\sqrt{5}]_B$ satisfies the condition

$$|5d^2 - n^2| = |3a - 2c| = 1\,.$$

Examples include

$$\frac{9}{4} \rightarrow [65, 72, 97]$$
$$\frac{38}{17} \rightarrow [1\,155, 1\,292, 1\,733]$$
$$\frac{161}{72} \rightarrow [20\,737, 23\,184, 31\,105]$$

[10]Erwin von Steinbach (c. 1244 – 17 January 1318) was a German architect, and was a central figure in the construction of the main facade of Strasbourg Cathedral. His sons Johannes and Gerlach, both architects, also worked on the cathedral.

The *Bautista family*[11] $[\sqrt{5}]_C$ satisfies the condition

$$|5d^2 - n^2| = |3a - 2c| = 4\,.$$

Examples include

$$\frac{7}{3} \rightarrow [40, 42, 58]$$
$$\frac{29}{13} \rightarrow [672, 754, 1\,010]$$
$$\frac{123}{55} \rightarrow [12\,104, 13\,530, 18\,154]$$

The *Wren family*[12] $[\sqrt{5}]_D$ satisfies the condition

$$|5d^2 - n^2| = |3a - 2c| = 5\,.$$

Examples include

$$\frac{5}{2} \rightarrow [21, 20, 29]$$
$$\frac{20}{9} \rightarrow [319, 360, 481]$$
$$\frac{85}{38} \rightarrow [5\,781, 6\,460, 8\,669]$$

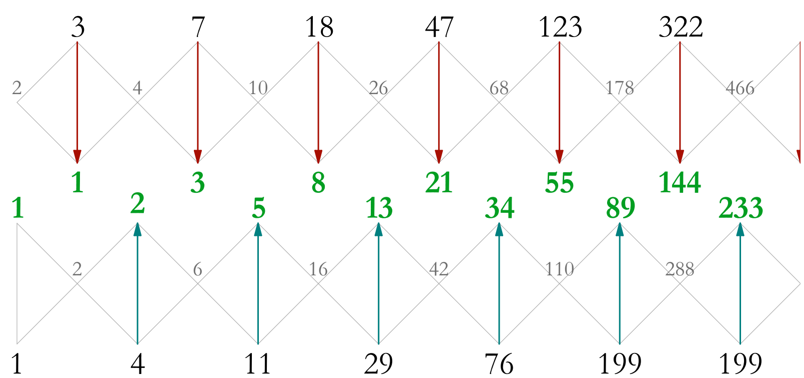

Figure 8: *Lattice of converging fractions of* $\sqrt{5}$

[11] Juan Bautista de Toledo (c. 1515 – 19 May 1567) is the Spanish architect who drew the plans the royal monastery San Lorenzo de El Escorial for King Philip II. He calls its underpinning geometry “Traza Universal” also known as the “Gridiron of St Lawrence”.

[12] Sir Christopher Wren PRS FRS (30 October 1632 – 8 March 1723) was one of the most highly acclaimed English architects in history. His masterpiece is Saint-Paul Cathedral, London.

You have probably noted that the $[\sqrt{5}]_A$ and $[\sqrt{5}]_C$ families both satisfy the condition $|3a-2c|=4$. Here is a neat way to unite them that I worked out using $\sqrt{5}=\varphi+\frac{1}{\varphi}$. As you can see in Figure 8, the two lattices showing the "upper bound" fractions and "lower bound" fractions of $\sqrt{5}$ are interfaced with the Fibonacci sequence. They show the same convergent fractions as $[\sqrt{5}]_A$, $[\sqrt{5}]_C$ and the double of $[\sqrt{5}]_B$. You also note that the upper and lower rows form the series of Lucas numbers[13] $[1,3,4,7,11,18,29,47,76,\ldots]$ whose successive ratios also converge towards φ.[14]

4.6 Copper families $[4,\overline{4}]$

Let $S_{[4,\overline{4}]}$ be the set of solutions t that defines the condition $|2b-a|=t$ for the four Pythagorean families of the Copper ratio $[4,\overline{4}]$. Then

$$S_{[4,\overline{4}]}=\{1,4,5\}\,.$$

The four Peacock diagrams of $[\sqrt{5}]$ in Figure 7 help visualise 16 sequences of fractions converging towards the "copper ratio". Indeed, for each of the four Peacock diagrams we have four parallel sequences satisfying

$$\lim_{x\to\infty}\frac{U_{n+4}}{U_n}=[4,\overline{4}]=\sqrt{5}+2\,.$$

As a result, each of the four Pythagorean families have a set of solutions t that defines its condition $|2b-a|=t$, as follows.

$[\sqrt{5}+2]_A$ has the set of solutions $S_A=\{4,16,20\}$.
Examples include

$$\begin{array}{lll}\frac{5}{1} \to & [24,10,26] & t=4\\ \frac{6}{2} \to & [32,24,40] & t=16\\ \frac{11}{3} \to & [112,66,130] & t=20\end{array}$$

[13] The Lucas numbers are an integer sequence named after the French mathematician François Édouard Anatole Lucas (1842–1891). He is the man who gave the name "Fibonacci Numbers" to the sequence written about by Leonardo of Pisa.

[14] The successive ratios arising from the Lucas numbers generate the *Lucas family* of Pythagorean triples where $\frac{|b-2a|}{2}=5$ and

$$\lim_{n\to+\infty}\frac{c_n}{a_n}=\lim_{n\to+\infty}\frac{2c_n}{b_n}=\sqrt{5}\,.$$

$[\sqrt{5}+2]_B$ has the set of solutions $S_B = \{1, 4, 5\}$.
Examples include

$$\begin{aligned} \frac{17}{4} &\to [273, 136, 305] & t = 1 \\ \frac{21}{5} &\to [416, 210, 466] & t = 4 \\ \frac{38}{9} &\to [1\,363, 684, 1\,525] & t = 5 \end{aligned}$$

$[\sqrt{5}+2]_C$ has the set of solutions $S_C = \{4, 16, 20\}$.
Examples include

$$\begin{aligned} \frac{13}{3} &\to [160, 78, 178] & t = 4 \\ \frac{29}{4} &\to [792, 406, 890] & t = 20 \\ \frac{42}{7} &\to [1\,664, 840, 1\,864] & t = 16 \end{aligned}$$

$[\sqrt{5}+2]_D$ has the set of solutions $S_D = \{5, 20, 25\}$.
Examples include

$$\begin{aligned} \frac{38}{9} &\to [1363, 684, 1525] & t = 5 \\ \frac{29}{7} &\to [792, 406, 890] & t = 20 \\ \frac{20}{5} &\to [375, 200, 425] & t = 25 \end{aligned}$$

As you can see, the four sets of solutions $S_{[\sqrt{5}+2]}$ are multiples of $S_{[4,\overline{4}]} = \{1, 4, 5\}$.

4.7 Bronze families $[3, \overline{3}]$

The bronze ratio defines three unique Peacock diagrams illustrated in Figure 9. Each diagram has three parallel sequences of fractions converging towards the bronze ratio:

$$\lim_{x \to \infty} \frac{U_{n+3}}{U_n} = [3, \overline{3}] = \frac{3+\sqrt{13}}{2} .$$

The resulting Pythagorean triples satisfy the condition

$$\frac{|3b - 2a|}{2} = t \quad \text{where} \quad t \in \{1, 3, 9\} .$$

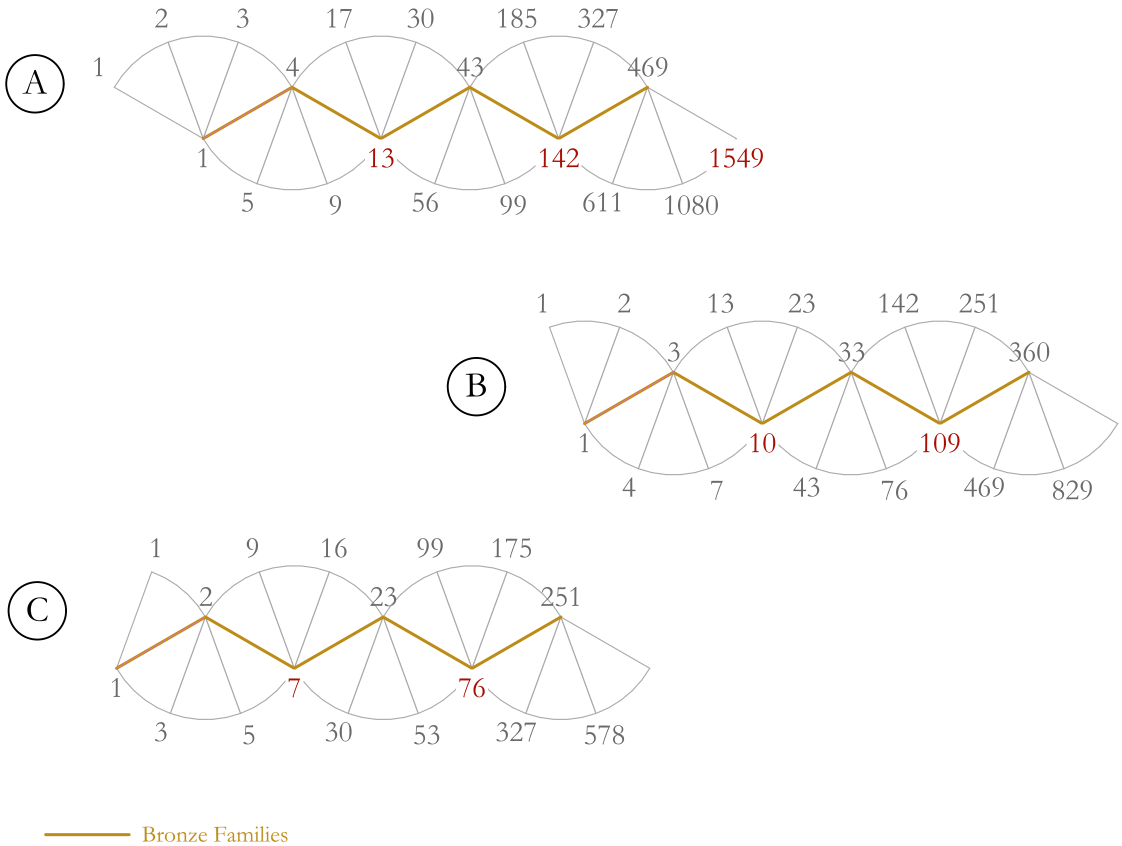

Figure 9: *The three peacock diagrams of the bronze ratio* $\frac{3+\sqrt{13}}{2}$

$[3, \overline{3}]_A$ family members satisfy the condition $\dfrac{|3b-2a|}{2} \in \{3, 9\}$.
Examples include

$$\begin{aligned}
\frac{4}{1} &\rightarrow [15, 8, 17] \qquad t = 3\\
\frac{13}{4} &\rightarrow [153, 104, 185] \quad t = 3\\
\frac{30}{9} &\rightarrow [819, 540, 981] \quad t = 9
\end{aligned}$$

$[3, \overline{3}]_B$ family members satisfy the condition $\dfrac{|3b-2a|}{2} \in \{1, 3\}$.
Examples include

$$\begin{aligned}
\frac{10}{3} &\rightarrow [91, 60, 109] \qquad t = 1\\
\frac{33}{10} &\rightarrow [989, 660, 1\,189] \quad t = 1\\
\frac{76}{23} &\rightarrow [5\,247, 3\,496, 6\,305] \quad t = 3
\end{aligned}$$

$[3,\overline{3}]_C$ family members satisfy the condition $\dfrac{|3b-2a|}{2} \in \{3,9\}$.
Examples include

$$\begin{array}{ccll} \frac{7}{2} & \to & [45,28,53] & t=3 \\ \frac{23}{7} & \to & [480,322,578] & t=3 \\ \frac{16}{5} & \to & [231,160,281] & t=9 \end{array}$$

Additional sequences of fractions converging towards the Bronze Ratio can be worked out from the ten Pythagorean families of $[\sqrt{13}] = [3;\overline{1,1,1,1,6}]$.

5 Categories of Pythagorean families

From the examples presented above, it seems that the infinite number of Pythagorean families can be sorted into three categories.

5.1 Algebraic families

Algebraic families are arising from sequences of fractions such as $\frac{xn+s}{yn+t}$.
The resulting triples $[a,b,c]$ have these types arithmetical conditions:

Pythagoras family :	$c-b=1$
Euclid family :	$c-b=4$
Plato family :	$c-a=2$
Socrates family :	$c-a=8$

5.2 Quadratic families

Quadratic families of $\sqrt{x} = [\frac{a_n}{2};\overline{a_1,a_2,\dots a_n}]$ arise from the various convergent sequences of fractions that emanate from its unique Peacock diagrams. Note that its Peacock diagrams are especially relevant since they will generate $a_1+\cdots+a_n$ sequences of converging fractions, whereas its periodic continued fractions will generate only n sequences.

Quadratic families show arithmetic connections between a and c:

Theon family $[\sqrt{2}]_A$	$c-3a=2$
Brouncker family $[\sqrt{2}]_B$	$c-3a=4$
Archimedes family $[\sqrt{3}]_A$	$c-2a=2$
Archimedes family $[\sqrt{3}]_B$	$c-2a=3$
Archimedes family $[\sqrt{3}]_C$	$2a-c=1$
Theodoros family $[\sqrt{5}]_A$	$\|3a-2c\|=4$
Steinbach family $[\sqrt{5}]_B$	$\|3a-2c\|=1$
Bautista family $[\sqrt{5}]_C$	$\|3a-2c\|=4$
Wren family $[\sqrt{5}]_D$	$\|3a-2c\|=5$

Theorem 1 (A Pell equation defines the condition of a family of Pythagorean triples.). *For any positive integer x such that $\sqrt{x} \notin \mathbb{Z}$, all members of a single $[\sqrt{x}]$ family of Pythagorean triples, where the sequence of converging fractions $\frac{n}{d}$ generates the triples*

$$[a, b, c] = [n^2 - d^2\,,\; 2nd\,,\; n^2 + d^2]\,,$$

satisfy the condition

$$|n^2 - xd^2| = \left|\frac{a(x+1) - c(x-1)}{2}\right| = t \quad \textit{where} \quad t \in \mathbb{N}\,.$$

An example

One of the Pythagorean families of $\sqrt{83} = [9, \overline{9, 18}]$ satisfies the condition

$$n^2 - 83d^2 = 42a - 41c = 1$$

which is a famous[15] Pell equation for $n = 83$. Here are its three first members:

$$\begin{aligned} \frac{82}{9} &\quad\to\quad [6\,643, 1\,476, 6\,805] \\ \frac{13\,447}{1\,476} &\quad\to\quad [178\,643\,233, 39\,695\,544, 183\,000\,385] \\ \frac{2\,205\,226}{242\,055} &\quad\to\quad [4\,804\,431\,088\,051, 1\,067\,571\,958\,860, 4\,921\,612\,334\,101] \end{aligned}$$

Let S_{83} be the set of solutions t for the equation $|n^2 - 83d^2| = |42a - 41c| = t$. Out of its 27 families of Pythagorean triples, the 15 possible solution t of S_{83} are[16]:

$$S_{83} = \{1, 17, 29, 37, 41, 83, 82, 79, 74, 67, 58, 47, 34, 19, 2\}\,.$$

Here are some of the minimal solutions:

$$\begin{aligned} n^2 - 83d^2 = 42a - 41c = 17 &\quad \text{is} \quad \frac{73}{8} \quad\to\quad [5\,265, 1\,168, 5\,393] \\ n^2 - 83d^2 = 42a - 41c = 29 &\quad \text{is} \quad \frac{64}{7} \quad\to\quad [4\,047, 896, 4\,145] \\ n^2 - 83d^2 = 42a - 41c = 37 &\quad \text{is} \quad \frac{55}{6} \quad\to\quad [2\,989, 660, 3\,061] \\ n^2 - 83d^2 = 42a - 41c = 41 &\quad \text{is} \quad \frac{46}{5} \quad\to\quad [2\,091, 460, 2\,141] \end{aligned}$$

[15] The solution $(82, 9)$ is the minimal solution to the Pell equation where $n = 83$. It was first identified by the famous Indian mathematician and astronomer Brahmagupta (598–668 CE).

[16] The order of the solutions in the set follows the individual trigons from the peacock diagram, when the initial couple of integers is $(1, 1)$.

Similarly, the 36 Pythagorean families of $\sqrt{61} = [7; \overline{1,4,3,1,2,2,1,3,4,1,14}]$ supply the set of 18 solutions[17] for $|n^2 - 61d^2| = |31a - 30c| = t$:

$$S_{61} = \{1, 12, 19, 20, 15, 3, 13, 4, 9, 12, 5, 25, 36, 45, 52, 57, 60, 64\}\,.$$

Other examples[18][19] include

$$\begin{aligned}
S_{13} &= \{1, 4, 3, 9, 12, 13\}\\
S_{29} &= \{1, 7, 4, 5, 13, 20, 25, 28, 29\}\\
S_{37} &= \{1, 12, 21, 28, 33, 36, 37\}\\
S_{63} &= \{1, 14, 27, 38, 47, 54, 59, 62, 63\}\\
S_{92} &= \{1, 11, 8, 13, 7, 19, 23, 4, 28, 43, 56, 67, 76, 83, 88, 91, 92\}\\
S_{101} &= \{1, 20, 37, 52, 55, 76, 85, 92, 97, 100, 101\}\\
S_{103} &= \{1, 18, 29, 34, 33, 26, 3, 13, 17, 6, 9, 11, 27, 39, 47, 51, 2, 22, 54, 67, 78, 87, 94, 99, 102, 103\}
\end{aligned}$$

5.3 Metallic families

Theorem 2 (Defining equation for metallic families of Pythagorean triples.)**.**
For any positive integer x, all members of a single $[x, \overline{x}]$ family of Pythagorean triples, where the sequence of converging fractions $\frac{n}{d}$ generates the triples

$$[a, b, c] = [n^2 - d^2\,,\ 2nd\,,\ n^2 + d^2]\,,$$

satisfy the following Pell-like condition

$$|xnd - n^2 + d^2| = \left|\frac{xb}{2} - a\right| = t \quad \textit{where} \quad t \in \mathbb{N}\,.$$

Examples of metallic families include

Golden family $[1, \overline{1}]$	$\lvert b - 2a\rvert = 2$	
Lucas family	$\lvert b - 2a\rvert = 10$	
Silver family $[2, \overline{2}]_A$	$\lvert b - a\rvert = 1$	*(a.k.a Fermat family)*
Silver family $[2, \overline{2}]_B$	$\lvert b - a\rvert = 2$	
Bronze families $[3, \overline{3}]$	$\left\lvert\frac{3b}{2} - a\right\rvert \in S_{[3,\overline{3}]}\{1, 3, 9\}$	
Copper families $[4, \overline{4}]$	$\lvert 2b - a\rvert \in S_{[4,\overline{4}]}\{1, 4, 5\}$	
Nickel families $[5, \overline{5}]$	$\left\lvert\frac{5b}{2} - a\right\rvert \in S_{[5,\overline{5}]}\{1, 5, 7\}$	

[17] The lowest solution for $n^2 - 61d^2 = 1$ was found by the Indian mathematician and astronomer Bhaskhara II in the 12th Century CE. The same equation was part of a challenge proposed by the French mathematician Pierre de Fermat. The English polymath Brouncker promptly found the solution.

[18] The Indian mathematician Brahmagupta found the minimal solution for $x^2 - 92y^2 = 1$ in 628CE.

[19] The Indian mathematician Narayana found the minimal solution for $x^2 - 103y^2 = 1$ during the 14th century.

For example, the condition on the Golden family of Pythagorean triples is

$$\left|nd - n^2 + d^2\right| = \left|\frac{b}{2} - a\right| = 1\,,$$

where d precedes n in the Fibonacci sequence; for example,

$$\frac{21}{13} \;\rightarrow\; [272, 546, 610]\,.$$

Indeed,

$$|21 \times 13 - 21^2 + 13^2| = \left|\frac{546}{2} - 272\right| = 1\,.$$

6 Generalised Pell equations

All of the sets S_x described above apply only when we use the initial couple of integers $(1,1)$ for any segment of the unique Peacock diagram of $\sqrt{x}$. The set can be infinitely expanded when we choose an alternative initial couple of integers.

Take for example the Peacock Diagram of $\sqrt{2}$, the set of solutions t for the initial couple $(1,1)$, for the condition $\frac{1}{2}(c - 3a) = t$ is $S_2 = \{1,2\}$ as we have seen from the Theon family and the Brouncker family of Pythagorean triples. Here are some alternative sets for various initial couples:

$$\begin{array}{lcl}
(1,1) & \rightarrow & \{1,2\} \\
(1,2) & \rightarrow & \{1,7\} \\
(1,3) & \rightarrow & \{2,17\} \\
(1,4) & \rightarrow & \{7,31\} \\
(1,5) & \rightarrow & \{14,49\} \\
(1,6) & \rightarrow & \{23,71\} \\
(2,7) & \rightarrow & \{17,94\}
\end{array}$$

The same applies to the above-mentioned equation for the metallic families of Pythagorean triples.

7 Khufu Challenge

Find the first three minimal solutions (n, d) for

$$n^2 - d^2 = 7nd + 673$$

and

$$n^2 - d^2 = 7nd - 673\,.$$

8 Proposal for a new Pythagorean tree

Pythagorean triples are the fruits of an arithmetical-lattice tree, and Pythagorean families arise from sequences of fractions which are the fruits of this tree. I therefore propose a new tree based on successive groups of three triangles; see Figure 10. There, each segment links the coprime numerator n and denominator d, where $n > d$, of a fraction $\frac{n}{d}$ which generates its own unique Pythagorean triple.

The initial segment of the tree unites the first numerator 2 and denominator 1. The value of the third vertex of the initial triangle is the sum of the two given vertex. It becomes numerator of two new fractions: $\frac{3}{1}$ and $\frac{3}{2}$. These two new fractions are the base of two new triangles and therefore four new fractions and their respective Pythagorean triples. The generated triples from this tree share a greatest common divisor which is either 1 or 2.

In this tree, a fraction $\frac{n}{d}$ generates the six following fractions:

$$F_R = \frac{n+d}{n} \quad F_{RR} = \frac{2n+d}{n} \quad F_{RL} = \frac{2n+d}{n+d} \quad F_L = \frac{n+d}{d} \quad F_{LL} = \frac{n+2d}{d} \quad F_{LR} = \frac{n+2d}{n+d}\,.$$

Similarly, its triple $[a, b, c]$ arising from the fraction $\frac{n}{d}$ generates the relevant six new Pythagorean triples thanks to the following matrix multiplicators:

$$M_R = \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} \quad M_{RL} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{3}{2} \\ 1 & 3 & 3 \\ \frac{3}{2} & 3 & \frac{7}{2} \end{pmatrix} \quad M_{RR} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

$$M_L = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 1 & 1 \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} \quad M_{LR} = \begin{pmatrix} -\frac{3}{2} & 1 & \frac{3}{2} \\ -1 & 3 & 3 \\ -\frac{3}{2} & 3 & \frac{7}{2} \end{pmatrix} \quad M_{LL} = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$$

For example, if we consider the fraction $\frac{5}{1}$ and its non-primitive triple $[10, 24, 26]$, it generates the six following fractions and their respective triples:

$$\begin{aligned} F_R &= \frac{6}{5} \;\rightarrow\; [11, 60, 61] & F_L &= \frac{6}{1} \;\rightarrow\; [12, 35, 37] \\ F_{RL} &= \frac{11}{6} \;\rightarrow\; [85, 132, 157] & F_{LR} &= \frac{7}{6} \;\rightarrow\; [13, 84, 85] \\ F_{RR} &= \frac{11}{5} \;\rightarrow\; [96, 110, 146] & F_{LL} &= \frac{7}{1} \;\rightarrow\; [14, 48, 50] \end{aligned}$$

When, as per the above example, n and d are odd, the fractions F_R, F_{RL}, F_L, F_{LR} generate primitive triples. When n is odd and d even, F_{RR}, F_{RL}, F_L, F_{LL} generate primitive triples. When n is even and d odd[20], F_R, F_{RR}, F_{LR}, F_{LL} generate primitive triples.

This slowly growing tree has the merit of visually highlighting the patterns of the various families presented. As a result, the successive fractions for each family can be traced as a route of successive *L*eft or *R*ight turns from the initial triangle.

[20] in this tree there cannot be a situation where n and d are both even

Examples include

Plato family	$[\overline{L}]$
Socrates family	$[\overline{R}]$
Archimedes$_A$ family $[\sqrt{3}]$	$[\overline{RL^2}]$
Archimedes$_C$ family $[\sqrt{3}]$	$[L\overline{RL^2}]$
Golden family $[1, \overline{1}]$	$[\overline{RL}]$

$\therefore$

"Wise men, Callicles, say that the heavens and the earth, gods and men, are bound together by fellowship and friendship, and order and temperance and justice, and for this reason they call the sum of ordered thing "Cosmos"[21]*, my friend, not disorder nor riot. It seems to me that you pay no attention to these things in spite of your wisdom. You are unaware that geometric equality is of great importance among gods and men alike, you think we should practice overreaching others, because you neglect geometry."* Gorgias $[508a]$, Plato.

$\because$

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[21] "Cosmos" is the word coined by Pythagoras to name the "ordered universe".

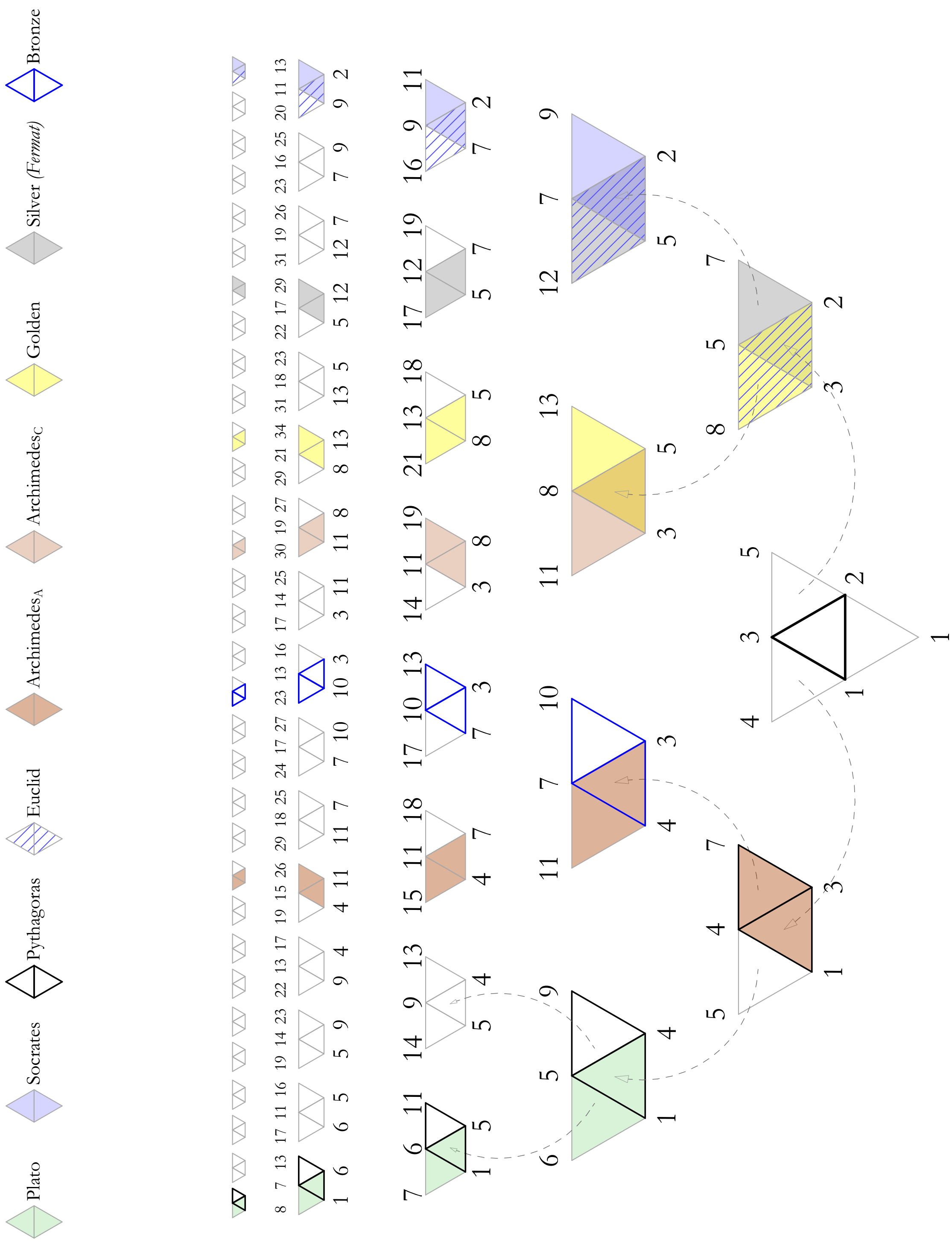

Figure 10: *Pythagorean tree highlighting some of the newly defined families*

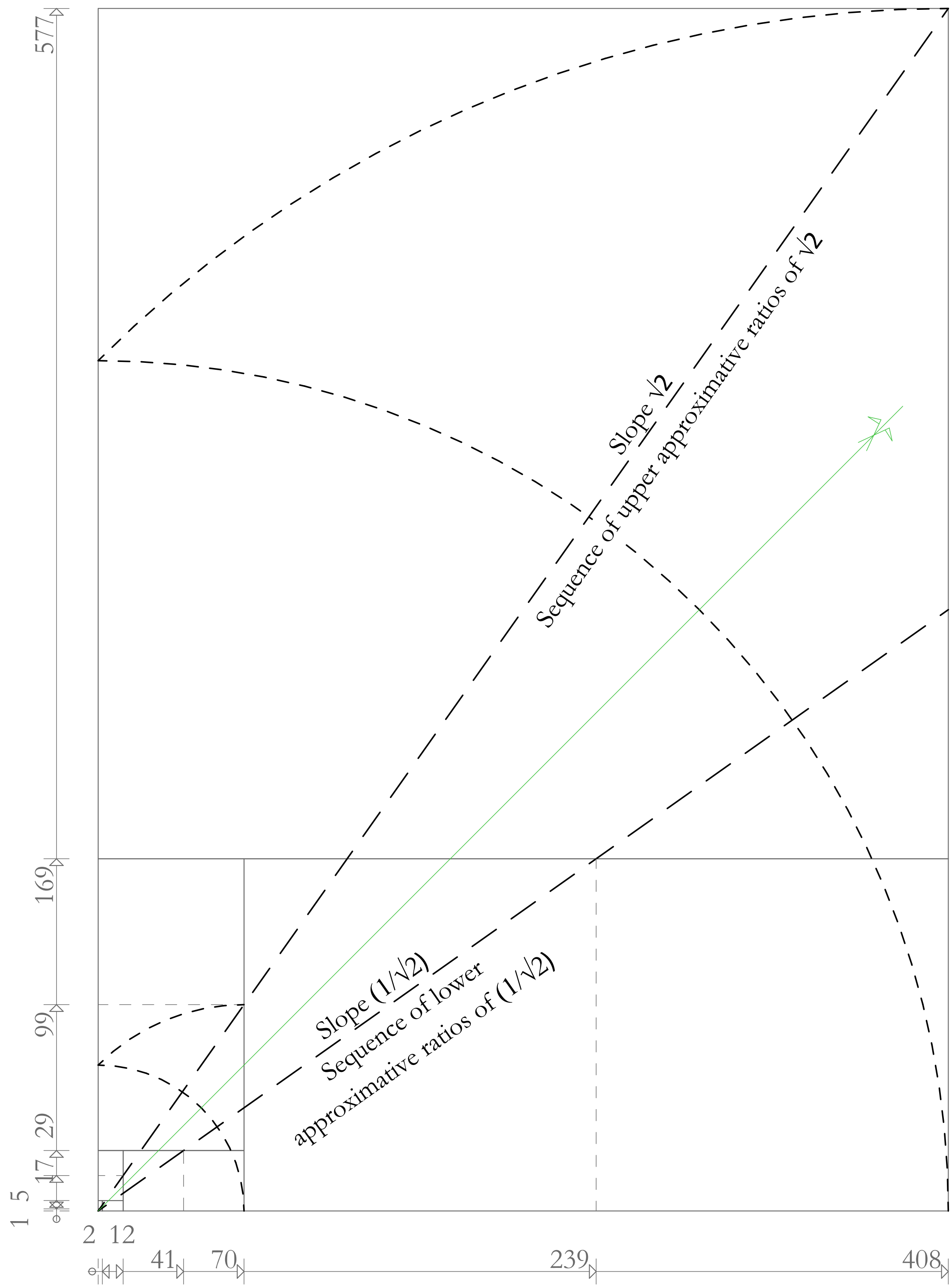

Figure 11: *Geometric algorithm for converging ratios of* $\sqrt{2}$ (Pell numbers)

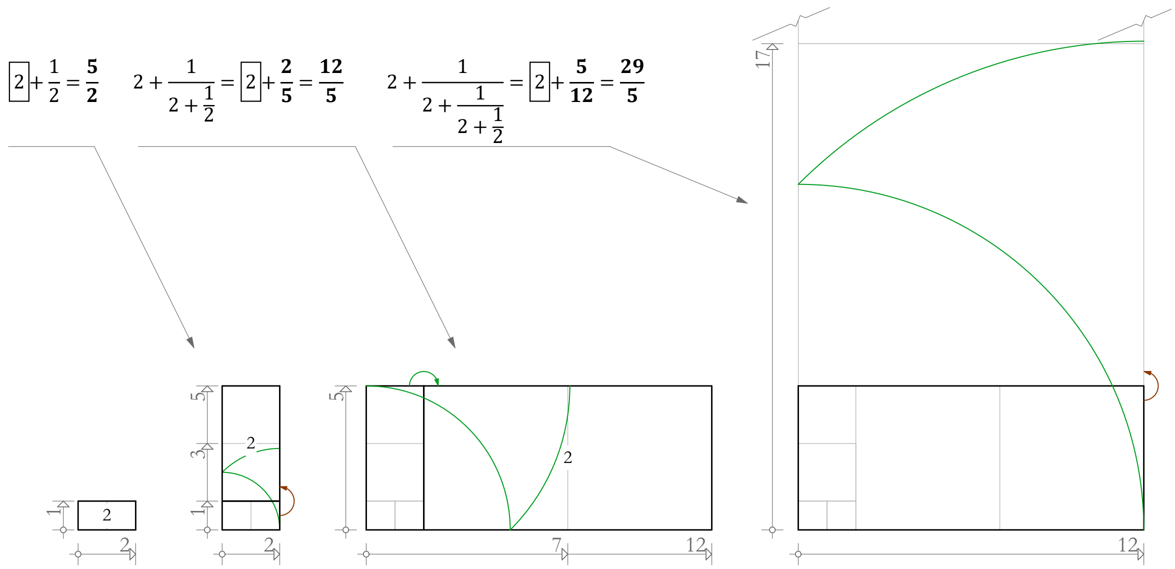

Figure 12: *Geometric algorithm for converging ratios of* $\sqrt{2}$ *(following steps on Figure 11)*

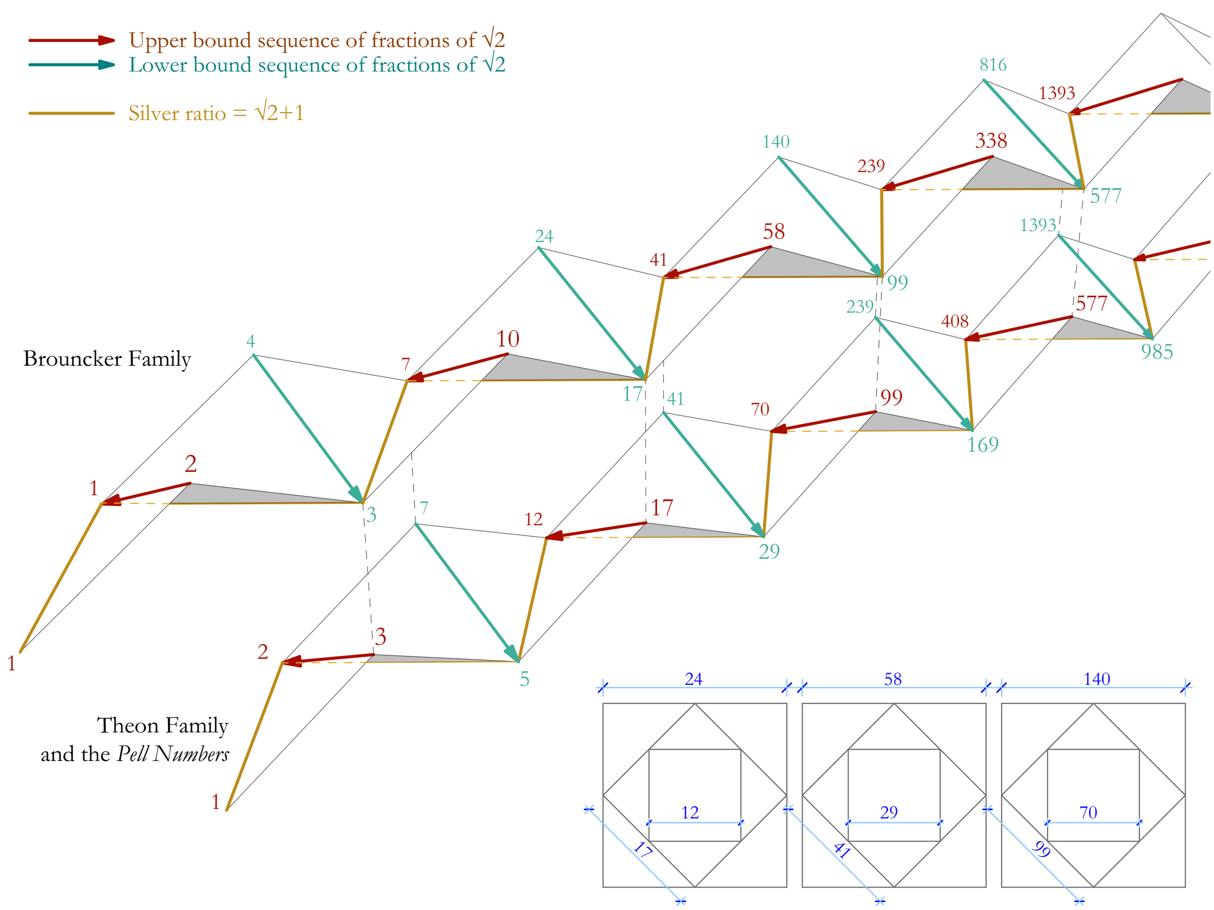

Figure 13: *3D lattice diagram and coprime nested squares showing converging ratios of* $\sqrt{2}$

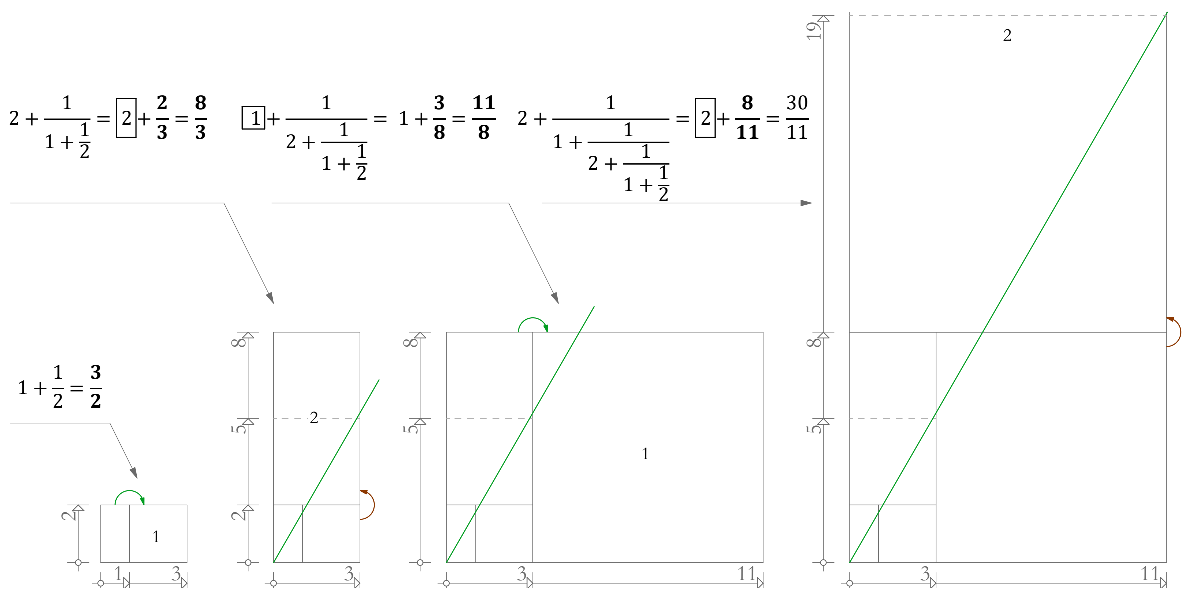

Figure 14: *Geometric algorithm for converging "lower bound" fractions of $\sqrt{3}$ (following steps on Figure 15)*

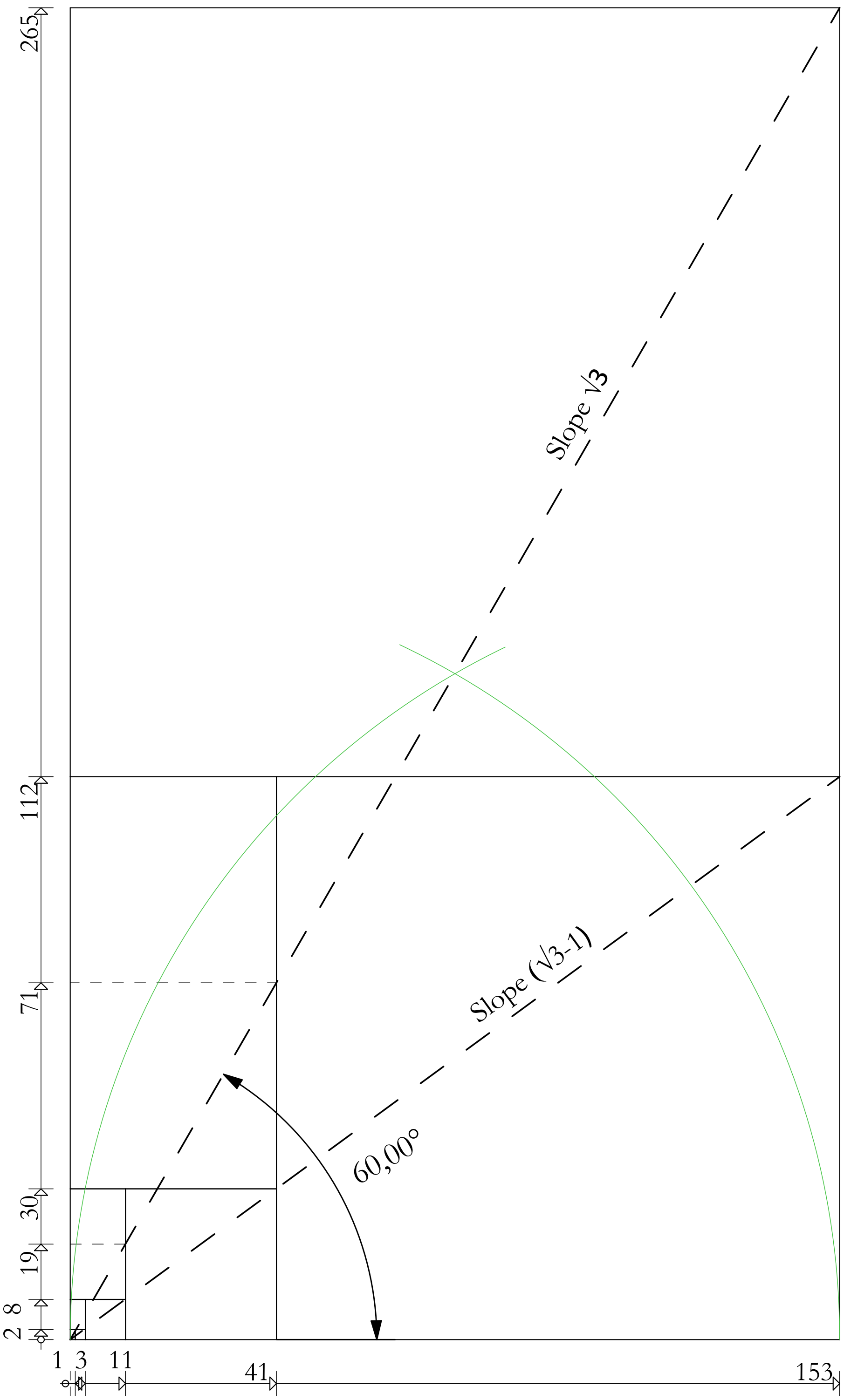

Figure 15: *Geometric algorithm for converging "lower bound ratios" of* $\sqrt{3}$

$$\boxed{1}+\frac{1}{3}=\frac{\mathbf{4}}{\mathbf{3}} \qquad \boxed{2}+\frac{1}{1+\frac{1}{3}}=2+\frac{\mathbf{3}}{\mathbf{4}}=\frac{\mathbf{11}}{\mathbf{4}} \qquad 1+\frac{1}{2+\frac{1}{1+\frac{1}{3}}}=\boxed{1}+\frac{\mathbf{4}}{\mathbf{11}}=\frac{15}{11}$$

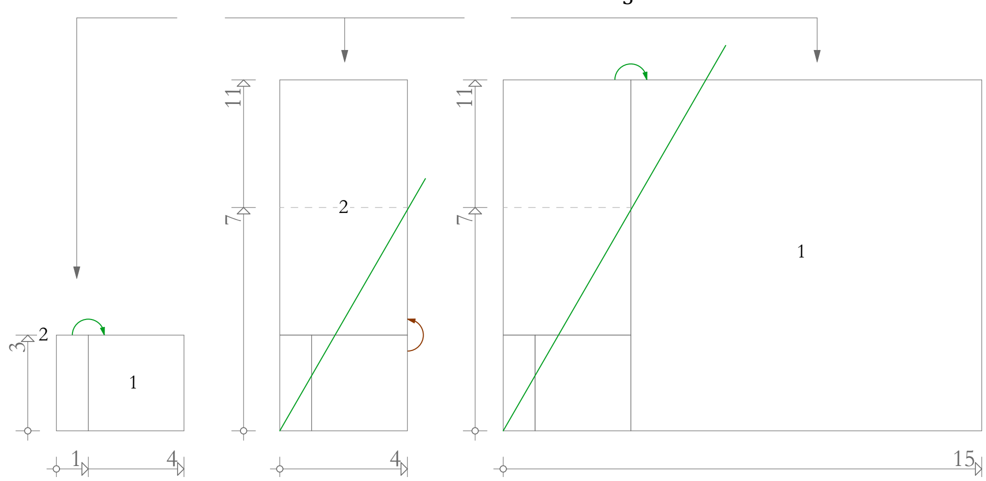

Figure 16: *Geometric algorithm for converging "upper bound" fractions of* $\sqrt{3}$ *(following steps on Figure 17)*

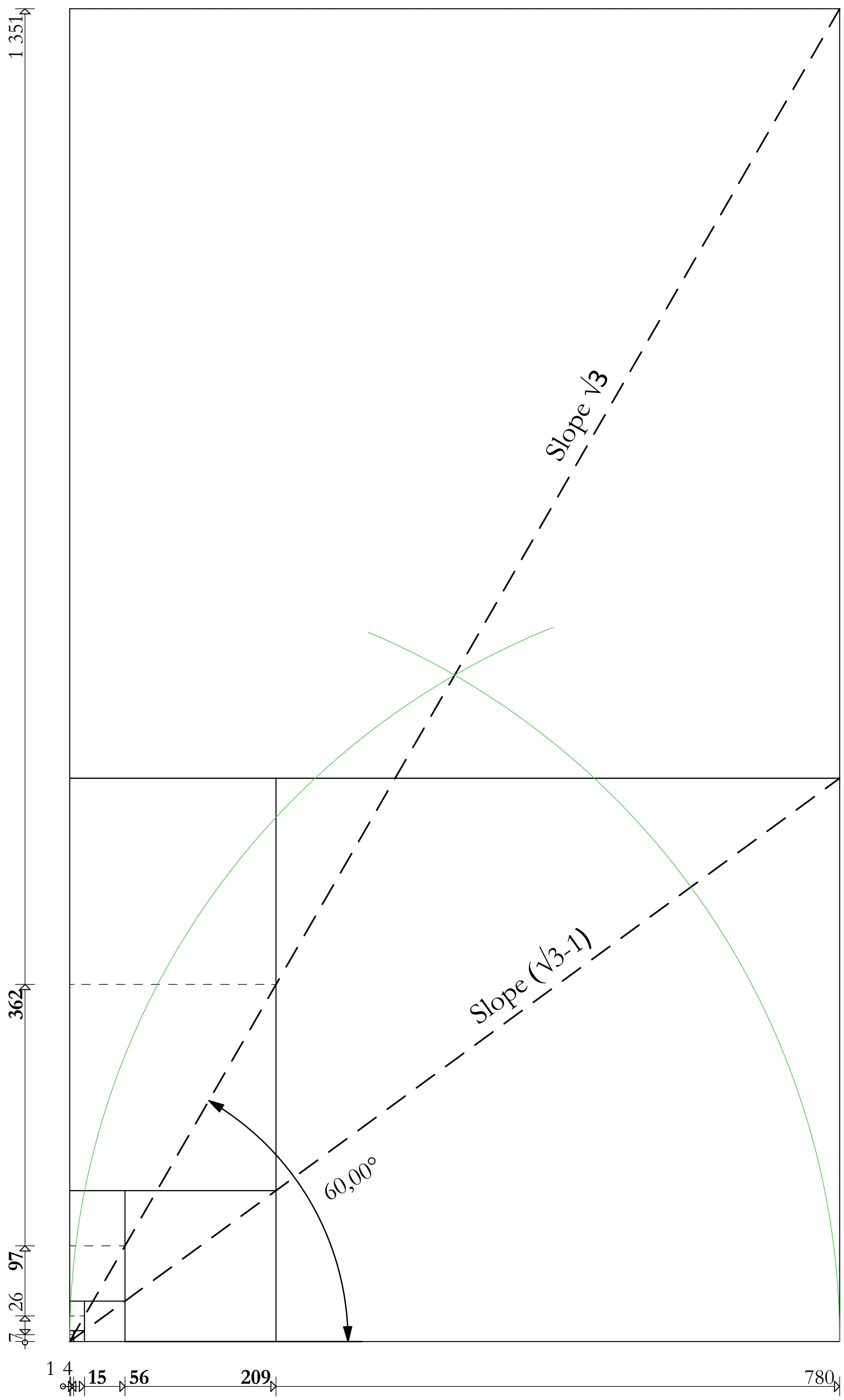

Figure 17: *Geometric algorithm for converging "upper bound ratios" of* $\sqrt{3}$

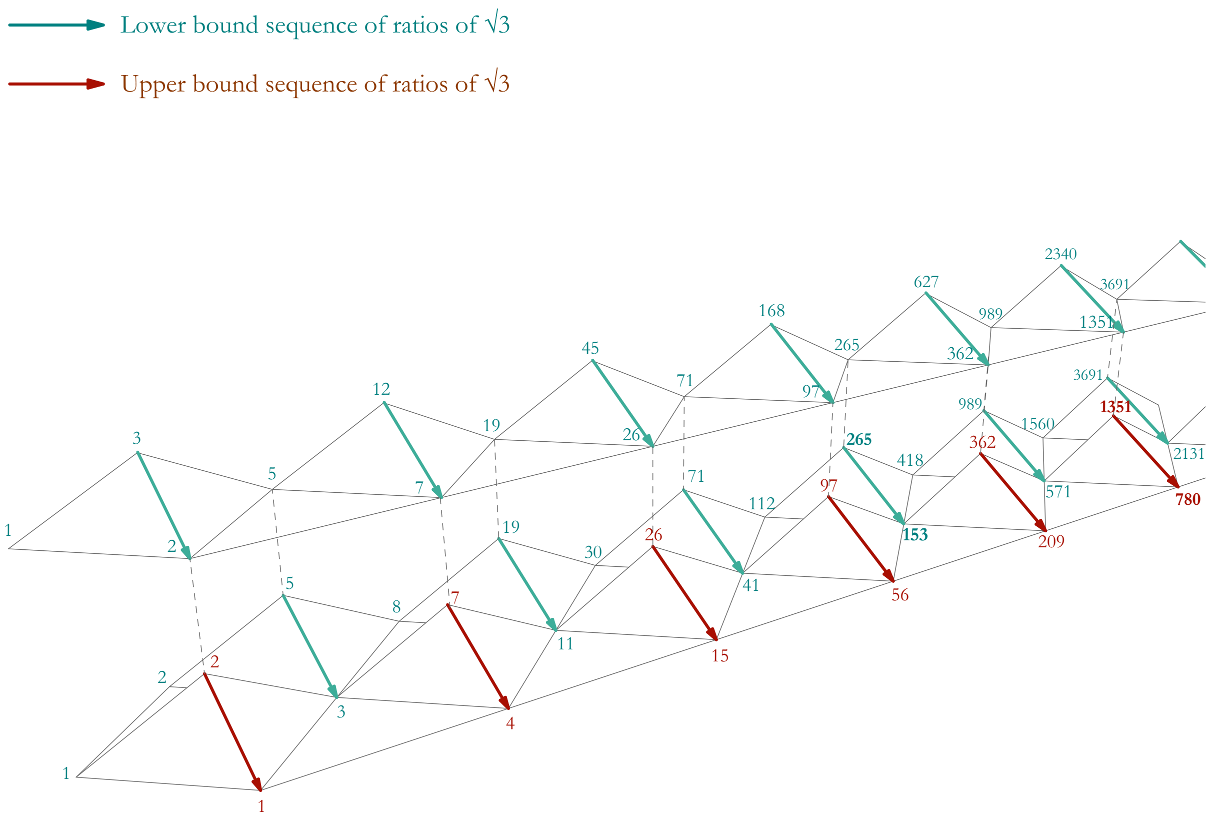

Figure 18: *3D lattice diagram of the three sequences of converging ratios of* $\sqrt{3}$

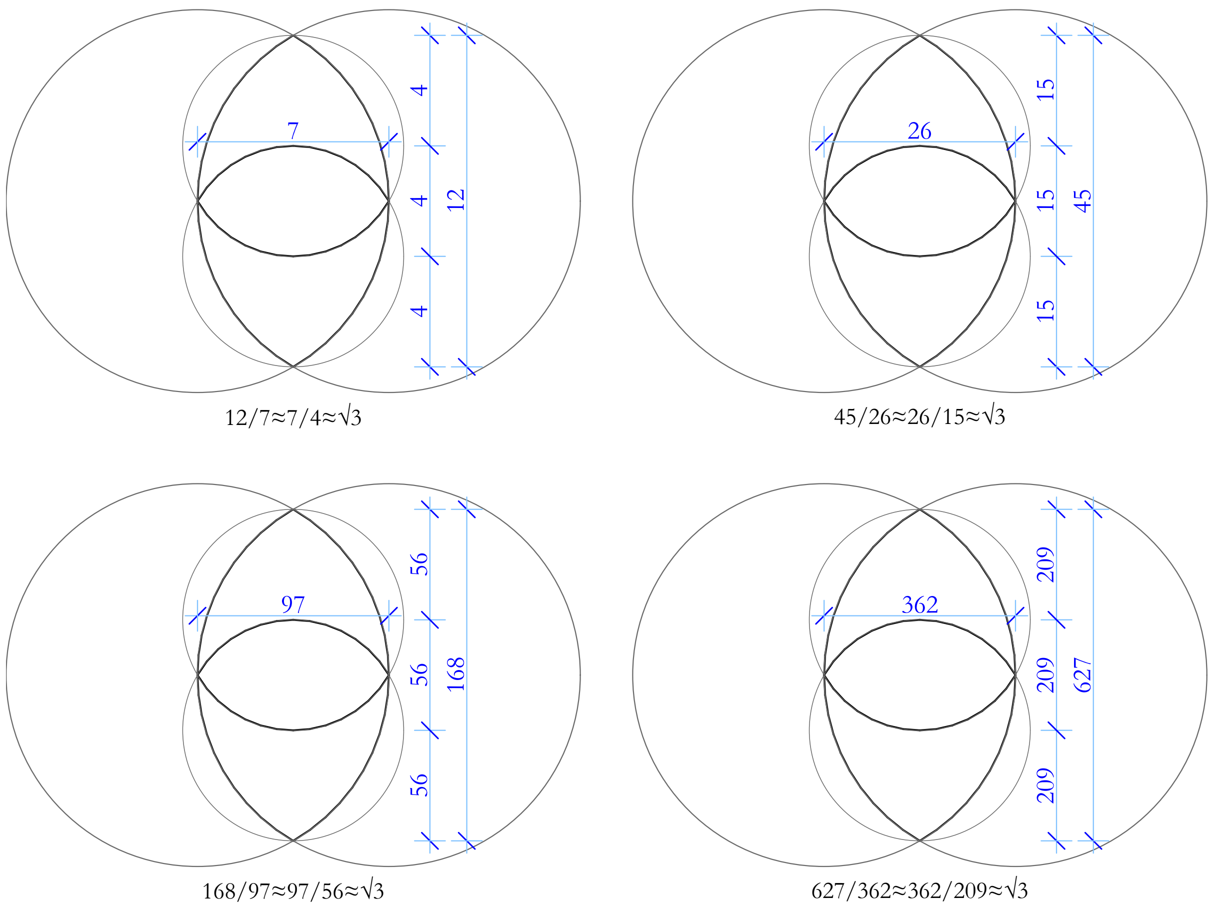

Figure 19: *Coprime nested rhombi showing converging ratios of* $\sqrt{3}$