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The beautiful inequality $\pi^e < e^\pi$

Kyumin Nam[1]

1 Introduction

Comparing π^e and e^π without a calculator is a problem that was known for a while. Also, this problem is in some mathematics books, too. This problem is not hard to solve, and if we solve this problem, then we obtain $\pi^e < e^\pi$.

Actually, we can calculate the values of π^e and e^π. Using a calculator, we obtain

$$\pi^e = 22.45915771836104547342715220454373502758931513399669224920$$
$$e^\pi = 23.14069263277926900572908636794854738026610624260021199344$$

which directly shows that $\pi^e < e^\pi$. However, in this paper, we'll introduce some proofs of the inequality $\pi^e < e^\pi$ and for more generalized inequalities, too.

2 Some proofs of the inequality $\pi^e < e^\pi$

First, we'll present some known proofs of the inequality $\pi^e < e^\pi$.

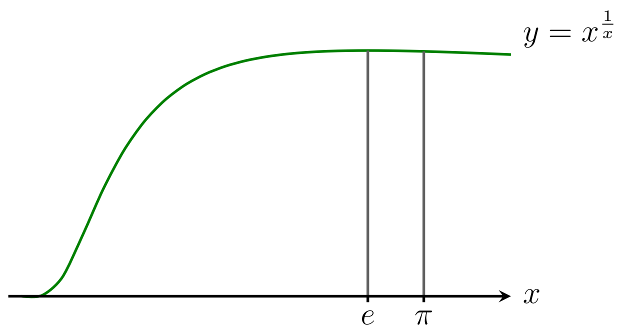
Theorem 1. $\pi^e < e^\pi$.

The following proof is well-known and is introduced as a solution in books containing this problem.

Proof. Consider the function $y = x^{\frac{1}{x}}$. Then, $y' = x^{\frac{1}{x}-2}(1 - \ln x)$ so function $y = x^{\frac{1}{x}}$ has maximum value at $x = e$. Thus,

$$e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$$

so $\pi^e < e^\pi$. □



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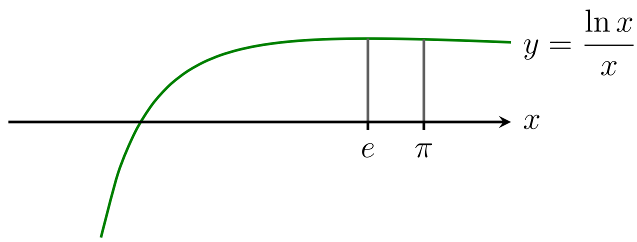
Similarly, we can consider $y = \frac{\ln x}{x}$ using same approach as above [5].

Proof. Consider the function $y = \dfrac{\ln x}{x}$. Then, $y' = \dfrac{1 - \ln x}{x^2}$ so function $y = \dfrac{\ln x}{x}$ has maximum value at $x = e$. Thus,

$$\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$$

so $\pi^e < e^\pi$. □

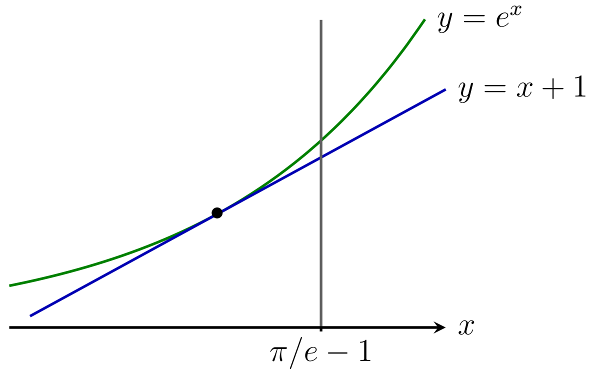


Also, we can use the fact that $e^x > x + 1$ for all $x > 0$.

Proof. Since $e^x > x + 1$ for all $x > 0$, letting $x = \dfrac{\pi}{e} - 1$ yields

$$e^{\frac{\pi}{e} - 1} > \frac{\pi}{e}$$

so $\pi^e < e^\pi$. □

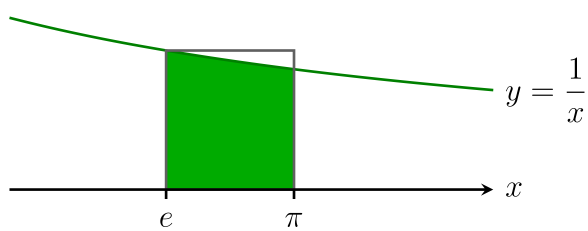


The next proof was found by Chakraborty [1] using an area argument.

Proof. The area of the rectangle consisting of points $(e, 0)$, $(\pi, 0)$, $(\pi, 1/e)$, $(e, 1/e)$ is greater than the area enclosed by the lines $y = 1/x$, $y = 0$, $x = e$, $x = \pi$. Therefore,

$$\ln \pi - 1 = \int_e^\pi \frac{dx}{x} < \frac{1}{e}(\pi - e) = \frac{\pi}{e} - 1$$

so $\pi^e < e^\pi$. □

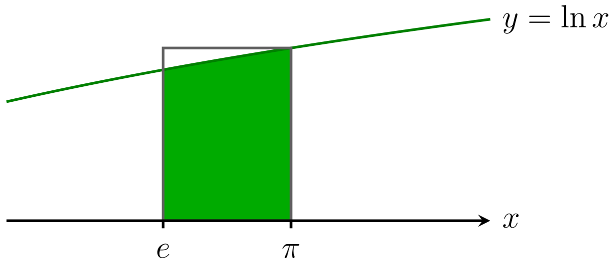


Using same approach with $y = \ln x$ yields the following proof from [2].

Proof. The area of the rectangle consist of points $(e, 0)$, $(\pi, 0)$, $(\pi, \ln \pi)$, and $(e, \ln \pi)$ is greater than the area enclosed by $y = \ln x$, $y = 0$, $x = e$, $x = \pi$. Therefore,

$$\pi \ln \pi - \pi = \int_e^\pi \ln x \, dx < \ln \pi \, (\pi - e) = \pi \ln b - e \ln \pi$$

so $\pi^e < e^\pi$. □

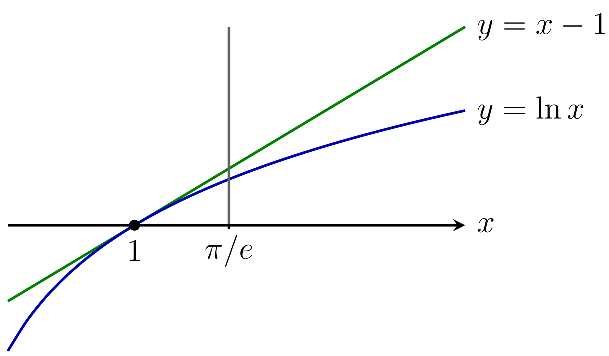


The next proof is by Chakraborty and Mukherjee [3] using a simple inequality that can be easily proved by elementary calculus.

Proof. Since $x - 1 > \ln x$ for all $x > 1$, letting $x = \dfrac{\pi}{e}$ yields

$$\ln \frac{\pi}{e} < \frac{\pi}{e} - 1$$

so $\pi^e < e^\pi$. □

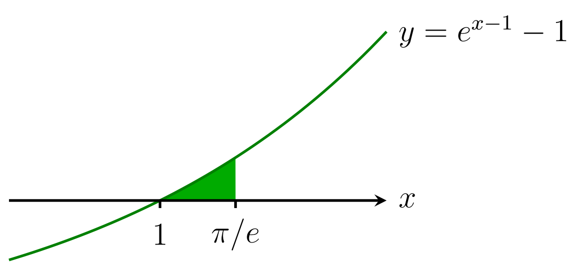


The next proof is found by Haque [4] also using a simple inequality that can be easily proved by elementary calculus.

Proof. Since $e^{x-1} - 1 > 0$ for all $x > 1$,

$$0 < \int_1^{\pi/e} \left(e^{x-1} - 1\right) \, dx = \frac{e^{\pi/e} - \pi}{e}$$

so $\pi^e < e^\pi$. □

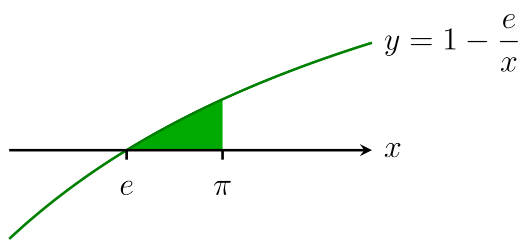


The following proofs from [8, 9, 11] follow a similar approach as the proofs above.

Proof. Since $1 - \frac{e}{x} > 0$ for all $x > e$,

$$0 < \int_e^\pi \left(1 - \frac{e}{x}\right) \mathrm{d}x = \pi - e \ln \pi$$

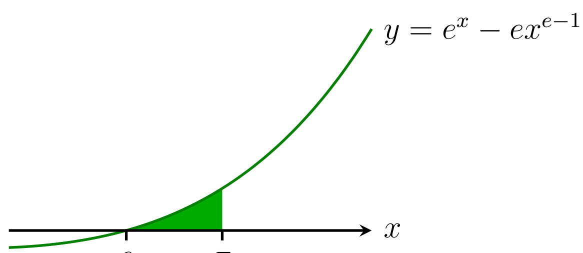
so $\pi^e < e^\pi$. $\square$



Proof. Since $e^x - ex^{e-1} > 0$ for all $x > e$,

$$0 < \int_e^\pi \left(e^x - ex^{e-1}\right) \mathrm{d}x = e^\pi - \pi^e$$

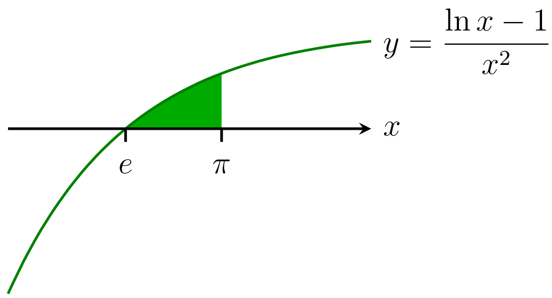
so $\pi^e < e^\pi$. $\square$



Proof. Since $\frac{\ln x - 1}{x^2} > 0$ for all $x > e$,

$$0 < \int_e^\pi \frac{\ln x - 1}{x^2} \mathrm{d}x = \frac{\ln e}{e} - \frac{\ln \pi}{\pi}$$

so $\pi^e < e^\pi$. $\square$



3 Generalisation of the inequality $\pi^e < e^\pi$

We can substitute π by any real number $b > e$ in the proofs introduced in Section 2, to prove the more generalized inequality $b^e < e^b$.

Theorem 2. *For all real numbers $b > e$, $b^e < e^b$.*

Proof. Substitute π by b in the proofs introduced in Section 2. □

Also, we can consider the most general inequality $b^a < a^b$ where $e < a < b$. Proof of this inequality can be completed by a suitable transformation of proofs in Section 2.

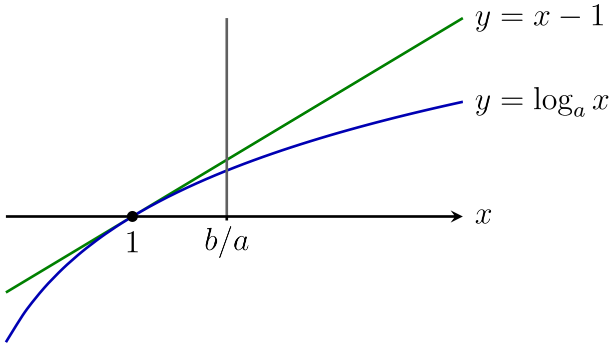
Theorem 3. *For all real numbers a and b with $e \le a < b$, $b^a < a^b$.*

The following proof from [10] is obtained by substituting $\log_a x$ where $e \le a$ for $\ln x$ in the proof of Chakraborty and Mukherjee [3].

Proof. Since $x - 1 \ge \log_a x$ for $a \ge e$, letting $x = \dfrac{b}{a}$ yields

$$\log_a \frac{b}{a} < \frac{b}{a} - 1$$

so $b^a < a^b$. □

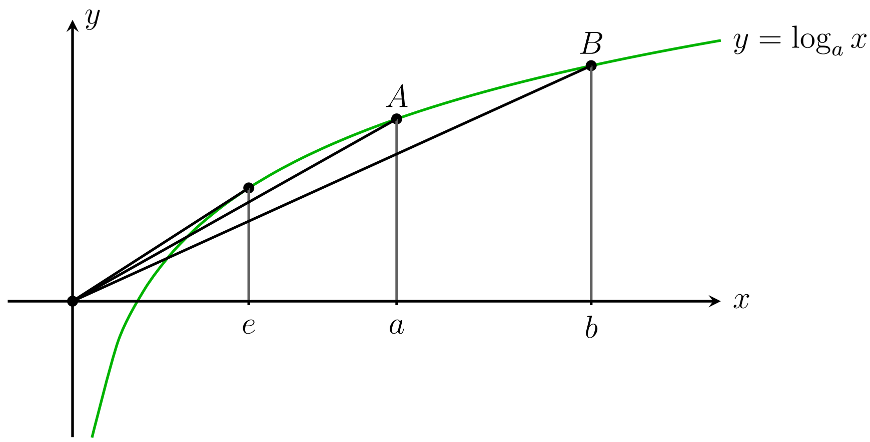


We now present a proof of Gallant [6] that appears to use almost the same approach as the second proof in Section 2 but actually uses a completely different approach.

Proof. Let m_A be the gradient of line $\overline{OA}$ where $O(0,0)$ and A is any point on function $y = \ln x$, and write $A(a, \ln a)$ and $B(b, \ln b)$ where $e \le a < b$. Then $m_A > m_B$, so

$$\frac{\ln a}{a} > \frac{\ln b}{b}.$$

It follows that $b^a < a^b$. □



4 Conclusion

In this article, we provided some known proofs of the beautiful inequality $\pi^e < e^\pi$. Also, we proved more general inequalities. There are several known proofs that have not been introduced in this article. But even though a lot of proofs are known, I have no doubt that there are a lot of undiscovered proofs and they are worth discussing, sharing and presenting.

Acknowledgements

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