Parabola Volume 59, Issue 2 (2023)

Calculating the number of side-based special right triangles

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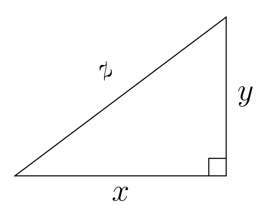
1 Introduction

A special right triangle is a right triangle whose angles or sides form a particular relationship. While an angle-based right triangle may have angles with a simple relationship, e.g. $45° - 45° - 90°$, a side-based right triangle is a special right triangle in which the lengths of its sides form ratios of whole numbers, e.g., $3 : 4 : 5$ [1]. The lengths of the sides of these special right triangles form Pythagorean triples [2]. In this paper, it is shown that, once the length of one of the legs in a right triangle is known, the total number of special right triangles in the form of Pythagorean triples can be found. In this work, a method based on the fundamental concepts of number theory is developed to find the number of these special right triangles with a fixed leg whose length is prespecified.

2 Method

Let $x \in \mathbb{Z}^+$ be the length of one of the legs of a right triangle. The length of the other leg is denoted as y, while z is the length of the hypotenuse as shown in the figure below.



By the Pythagorean theorem, these numbers satisfy the following equation:

$$x^2 + y^2 = z^2 .$$

The square of the length x can be factorized as

$$x^2 = z^2 - y^2 = (z - y)(z + y) .$$

The sum of the two factors of x^2, which are $(z - y)$ and $(z + y)$, is equal to two times the length of the hypotenuse:

$$(z - y) + (z + y) = 2z .$$

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Using a factor pair (a, b) of x^2, one can define a right triangle with a leg of length x and a hypotenuse of length z. Given $x^2 \in \mathbb{Z}^+$, there exist $a, b \in \mathbb{Z}^+$ with $b > a$ such that

$$x^2 = (z - y)(z + y) = ab\,.$$

Thus, the length of the hypotenuse of the triangle can be calculated as

$$z = \frac{a + b}{2}\,.$$

We can also determine the length of the other leg of the triangle:

$$y = \sqrt{z^2 - x^2} = \frac{b - a}{2}\,.$$

A Pythagorean triple, denoted by (x, y, z), consists of positive integers x, y, and z satisfying the Pythagorean equation [2]. Here, given the value of x, we provide the following two criteria for generating (finding) the special right triangles in the form of Pythagorean triples (x, y, z), where the lengths of the other leg and hypotenuse are $y \in \mathbb{Z}^+$ and $z \in \mathbb{Z}^+$, respectively.

One can find infinitely many pairs (a, b) such that $x^2 = ab$, when a and b can be any positive real number. Therefore, there exist infinitely many right triangles whose one leg has a length of x. However, if the right triangles are restricted to be special, i.e., y and z are both positive integers, then the number of special right triangles that can be found becomes finite.

Criterion 1. *y and z must both be positive integers.*

To find y and z, one can use the factor pairs of positive integers (a, b) for x^2. To find a candidate length z of the hypotenuse, consider a factor pair (a, b) of x^2: $x^2 = ab$. If $a + b$ is not even, then $z = (a + b)/2$ is not an integer. This gives us the second criterion.

Criterion 2. *Given x, for a special right triangle generated by a factor pair (a, b) of x^2, the sum of the elements of the factor pair, $a + b$, must be even.*

Based on the criteria above, given the length x of one leg, we can generate distinct special right triangles using the factor pairs (a, b) of x^2, $b > a$, provided that the sum of the elements of each pair is even.

First, we recall a well-known method to find the number of divisors of an integer [2]. Let n be a positive integer, $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers. Then, the number of divisors of n is $d = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1)$. Suppose that

$$x = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}\,,$$

where $p_1, p_2, \ldots, p_r$ are distinct prime numbers. Then

$$x^2 = p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}\,.$$

While the number of divisors of x^2, $D = (2a_1 + 1)(2a_2 + 1) \cdots (2a_r + 1)$, is an odd number, the number of factor pairs of x^2, excluding the pair (x, x), is equal to $(D - 1)/2$.

Here, we develop methods for determining the special right triangles and, thus, for finding the number of these triangles depending on the parity of x. We also explain how the procedures differ when x is a power of 2 and when it is not, in the case when x is even.

Case 1: x is odd

If x, the given length of one leg, is odd, then x^2 is also odd. Therefore, all of the divisors of x^2 will be odd integers. That implies that the sum of any factor pair (a, b) of x^2 is always even. Satisfying both Criteria 1 and 2, the number of special right triangles that can be generated is equal to $(D-1)/2$, where D is the number of divisors of x^2.

Write

$$x = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

where $p_1, p_2, \ldots, p_r$ are distinct odd prime numbers. Then the number of special right triangles, N_{SRT}, that exist (can be generated) is as follows:

$$N_{SRT} = \frac{D-1}{2} = \frac{(2a_1+1)(2a_2+1)\cdots(2a_k+1) \ - \ 1}{2} = \frac{1}{2}\left[\left(\prod_{i=1}^{k}(2a_i+1)\right) - 1\right].$$

Example 1. Suppose that the length of one triangle leg is $x = 105 = 3 \times 5 \times 7$. Then

$$x^2 = 3^2 \times 5^2 \times 7^2 .$$

The number of special right triangles is therefore $N_{SRT} = \big((2 \times 1 + 1)^3 - 1\big)/2 = 13$. The factor pairs (a, b) of $x^2 = ab$ and the list of the Pythagorean triples defining these special right triangles are given in Table 1.

No.	a	b	(x, y, z)
1	1	11025	$(105, 5512, 5513)$
2	3	3675	$(105, 1836, 1839)$
3	5	2205	$(105, 1100, 1105)$
4	7	1575	$(105, 784, 791)$
5	9	1225	$(105, 608, 617)$
6	15	735	$(105, 360, 375)$
7	21	525	$(105, 252, 273)$
8	25	441	$(105, 208, 233)$
9	35	315	$(105, 140, 175)$
10	45	245	$(105, 100, 145)$
11	49	225	$(105, 88, 137)$
12	63	175	$(105, 56, 119)$
13	75	147	$(105, 36, 111)$

Table 1: Pythagorean triples for $x = 105$ (Example 1)

Case 2: x is even

If x is even, then x^2 is also even. In each factor pair of an even number, one of the elements must be even, while the other could be even or odd.

Case 2a.

Suppose that

$$x = 2^m$$

is a power of two, where $m \in \mathbb{Z}^+$. Then any of the divisors of x is also a power of 2. That implies that any factor pair (a, b) of x^2 includes no odd number except for 1. Therefore, the sum of any factor pair, $a + b$, is even except for the pair that includes 1.

By Criterion 2, given x as the length of one triangle leg, each of these factor pairs, except for the pair (x, x), defines a special right triangle. The number of these factor pairs that do not include 1 is then

$$\frac{(2m+1)-1}{2} - 1 = m - 1\,.$$

This is the number of special right triangles that can be found:

$$N_{SRT} = m - 1\,.$$

Example 2. Suppose that the length of one triangle leg is

$$x = 2048 = 2^{11}\,.$$

The number of special right triangles is then $N_{SRT} = 11 - 1 = 10$. The factor pairs (a, b) of $x^2 = ab$ and the list of the Pythagorean triples defining these special right triangles are given in Table 2.

No.	a	b	(x, y, z)
1	2	2097152	$(2048, 1048575, 1048577)$
2	4	1048576	$(2048, 524286, 524290)$
3	8	524288	$(2048, 262140, 262148)$
4	16	262144	$(2048, 131064, 131080)$
5	32	131072	$(2048, 65520, 65552)$
6	64	65536	$(2048, 32736, 32800)$
7	128	32768	$(2048, 16320, 16448)$
8	256	16384	$(2048, 8064, 8320)$
9	512	8192	$(2048, 3840, 4352)$
10	1024	4096	$(2048, 1536, 2560)$

Table 2: Pythagorean triples for $x = 2048$ (Example 2)

Case 2b.

Suppose now that x has both even and odd divisors. Then x^2 also has both even and odd divisors. Some factor pairs of x^2 therefore include both an even and an odd number and the sum of such a pair is odd. By Criterion 2, these factor pairs do not define a special right triangle and must be ignored when counting the factor pairs for special right triangles. To find the factor pairs of x^2 having an odd number in them, we need to find all the divisors of x^2 that are odd. Write

$$x = 2^{a_2} p_2^{a_2} \cdots p_k^{a_k}$$

where $p_2, \ldots, p_k$ are distinct odd prime numbers. The number of odd divisors of x^2 equals the number of divisors of

$$p_2^{2a_2} \cdots p_k^{2a_k} ,$$

namely

$$(2a_2 + 1) \cdots (2a_k + 1) = \prod_{i=2}^{k} (2a_i + 1) .$$

The total number of factor pairs of x^2, excluding the pair (x, x), is

$$\frac{(2a_1 + 1)(2a_2 + 1) \cdots (2a_k + 1) - 1}{2} = \frac{1}{2} \left[\left(\prod_{i=1}^{k} (2a_i + 1) \right) - 1 \right] .$$

The number of special right triangles is then the difference between these two numbers:

$$N_{SRT} = \frac{1}{2} \left[\left(\prod_{i=1}^{k} (2a_i + 1) \right) - 1 \right] - \prod_{i=2}^{k} (2a_i + 1) = \frac{1}{2} \left[\left((2a_1 - 1) \prod_{i=2}^{k} (2a_i + 1) \right) - 1 \right] .$$

Example 3. Suppose that the length of one triangle leg is

$$x = 728 = 2^3 \times 7 \times 13 .$$

Then $(a_1, a_2, a_3) = (3, 1, 1)$, so the number of special right triangles is

$$N_{SRT} = \frac{1}{2} \big((2 \times 3 - 1)(2 \times 1 + 1)(2 \times 1 + 1) - 1 \big) = 22 .$$

The factor pairs (a, b) of $x^2 = ab$ and the list of the Pythagorean triples defining these special right triangles are given in Table 3.

3 Conclusion

The number of the side-based special right triangles that have a leg of prespecified length is finite. The number of these triangles can easily be found by the method developed in this study. The method covers all of the possible cases depending on the parity and the prime factors of the given leg length of the special triangles that can be generated.

No.	a	b	(x, y, z)
1	2	264992	$(728, 132495, 132497)$
2	4	132496	$(728, 66246, 66250)$
3	8	66248	$(728, 33120, 33128)$
4	14	37856	$(728, 18921, 18935)$
5	16	33124	$(728, 16554, 16570)$
6	26	20384	$(728, 10179, 10205)$
7	28	18928	$(728, 9450, 9478)$
8	32	16562	$(728, 8265, 8297)$
9	52	10192	$(728, 5070, 5122)$
10	56	9464	$(728, 4704, 4760)$
11	98	5408	$(728, 2655, 2753)$
12	104	5096	$(728, 2496, 2600)$
13	112	4732	$(728, 2310, 2422)$
14	182	2912	$(728, 1365, 1547)$
15	196	2704	$(728, 1254, 1450)$
16	208	2548	$(728, 1170, 1378)$
17	224	2366	$(728, 1071, 1295)$
18	338	1568	$(728, 615, 953)$
19	364	1456	$(728, 546, 910)$
20	392	1352	$(728, 480, 872)$
21	416	1274	$(728, 429, 845)$
22	676	784	$(728, 54, 730)$

Table 3: Pythagorean triples for $x = 728$ (Example 3)

Acknowledgements

I would like to thank my teachers Ms. Derya Haseki and Mr. Suheyl Dogan for their insightful suggestions and continuous support during this study.

References

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