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Cups made from a square sheet of paper

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1 Introduction

We study a problem in which we make a cup with a square sheet of paper and try to make the volume of the cup as big as possible. When the shape of the cup is a rectangular parallelepiped, this problem is an easy exercise of elementary calculus, but when the shape is a square frustum, this problem becomes difficult to solve.

Two of the present authors published part of this paper as [1] in Japan, but in the present paper, we solve the same problem by a method different from that used in [1].

We start with a well-known exercise problem from elementary calculus, and we generalize the problem to make a new problem.

2 Cups made from a square paper

Problem 1. *We have a square sheet of paper of* 12*cm* $\times$ 12*cm as in Figure 1, and we are going to cut out the corners of the paper as in Figure 2 and fold the sides to form a cup as in Figure 3. Find the height of the box that will give the maximal volume.*

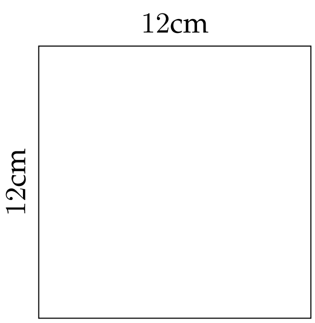

Figure 1

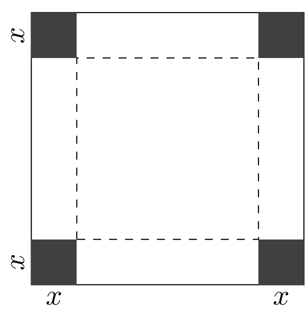

Figure 2

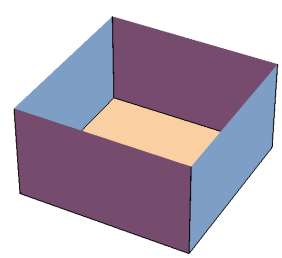

Figure 3

[1]Students at Keimei Gakuin High School in Japan

[2]Daisuke Ikeda is an independent researcher in Japan.

[3]Ryohei Miyadera, Ph.D., is a mathematician and mathematics advisor at Keimei Gakuin in Japan.

Solution. If the side of the square to be cut out has length x, then, after we cut out squares from each corner, the side length of the base of the box is $12 - 2x$. The volume of the box is therefore

$$V = x(12 - 2x)^2 \,. \tag{1}$$

Since $\frac{dV}{dx} = 12x^2 - 96x + 144 = 12(x-6)(x-2) = 0$ and $x < 6$, we have $x = 2$, and hence the maximum value of V is 128 when $x = 2$. See Figure 4 for the graph of V.

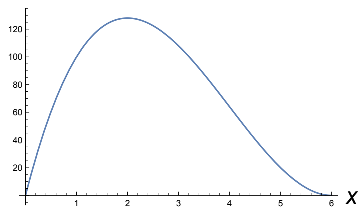

Figure 4: $V = x(12 - 2x)^2$

When Ryohei Miyadera, one of the authors, taught Problem 1 in a high school classroom, he asked his students to try to change some parts of the problem. Daisuke Ikeda, also one of the authors, was in the classroom as a student. He proposed the following Problem 2, which was quite challenging to solve.

One of the most common mathematical research methods is to generalize the original problems to make new ones. D. Ikeda's idea was a very natural generalization of Problem 1.

Problem 2. *We have a square sheet of paper of* 12*cm* $\times$ 12*cm as in Figure 1, and we are going to cut out corners of the paper as in Figure 5 and fold the sides to form a cup of a square frustum as in Figure 6. Find* x *and* y *that will give the maximal volume.*

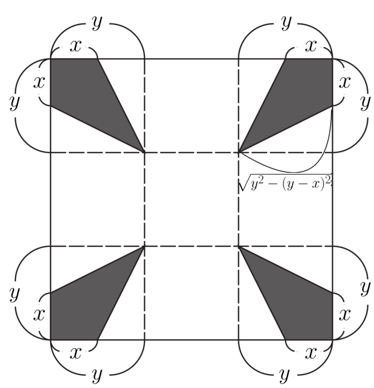

Figure 5

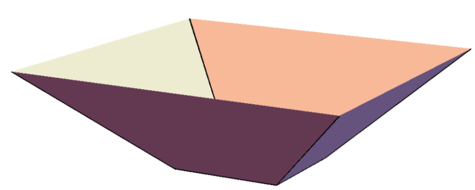
Figure 6

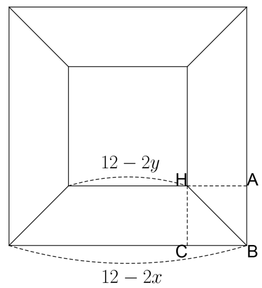

Figure 7

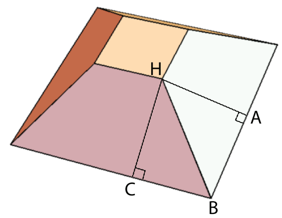

Figure 8

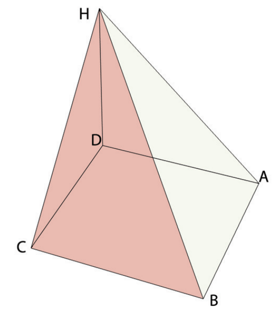

Figure 9

Solution Figure 7 shows the square frustum in Figure 8 from above, and Figure 9 is a part of the square frustum in Figure 8. By Figure 7, we have

$$AB = CB = y - x\,. \tag{2}$$

Next, we consider the shape in Figure 9. By (2), $BD = \sqrt{2}(y-x)$. By Figure 5, $BH = \sqrt{y^2 + (y-x)^2}$, and hence the height is

$$DH = \sqrt{BH^2 - BD^2} = \sqrt{y^2 + (y-x)^2 - 2(y-x)^2} = \sqrt{y^2 - (y-x)^2}\,. \tag{3}$$

We need the following lemma to calculate the volume. For a proof, see [2].

Lemma 1. *For a square frustum, let h be the height, A_1 the bottom area, and A_2 the top area. Then the volume of the pyramidal frustum is given by $\frac{h(A_1+A_2+\sqrt{A_1A_2}}{3}$.*

Next, we calculate the volume of the cup in Figure 6. The top area and the bottom area are $(12-2y)^2$ and $(12-2x)^2$ respectively, and hence by (3) and Lemma 1, we have

$$\begin{aligned} V &= \frac{\sqrt{y^2-(y-x)^2}((12-2y)^2 + (12-2x)^2 + (12-2y)(12-2x))}{3} \\ &= \frac{4}{3}\left(x^2 + xy - 18x - 18y + y^2 + 108\right)\sqrt{y^2-(y-x)^2}\,. \end{aligned} \tag{4}$$

If you want to find the maximum volume of V by numerical methods, then you can use mathematical software. See Example 1 in the Appendix.

We will find x and y that give the maximal volume. Since V is a continuous function of x and y, we look for the points where $\frac{dV}{dx} = \frac{dV}{dy} = 0$ to find x and y that make V maximal. By (4), we have

$$\frac{dV}{dx} = -\frac{4\big(3x^3 - 3x^2(y+12) - 2x(y^2-18y-54) - y(y^2-18y+108)\big)}{3\sqrt{-x(x-2y)}} = 0 \tag{5}$$

and

$$\frac{dV}{dy} = \frac{4x\big((x-54)y + 5y^2 + 108\big)}{3\sqrt{-x(x-2y)}} = 0. \tag{6}$$

By (5) and (6) we have

$$3x^3 - 3x^2(y+12) - 2x(y^2 - 18y - 54) - (y^3 - 18y^2 + 108y) = 0 \tag{7}$$

and

$$(x-54)y + 5y^2 + 108 = 0\,. \tag{8}$$

By (8) we have

$$x = \frac{-5y^2 + 54y - 108}{y}\,, \tag{9}$$

and hence by (7),

$$49y^6 - 1400y^5 + 16284y^4 - 98928y^3 + 331776y^2 - 583200y + 419904 = 0\,,$$

which can also be written as

$$(y-6)(7y^2 - 54y + 108)(7y^3 - 104y^2 + 468y - 648) = 0\,.$$

Since $y < 6$ and the solutions of $7y^2 - 54y + 108 = 0$ are not real numbers; we only have to solve

$$7y^3 - 104y^2 + 468y - 648 = 0\,. \tag{10}$$

If we use mathematics software such as Mathematica, then we get the numerical solutions $y = 2.7861, 4.2461, 7.8248$. See Example 2 in the Appendix.

Since $y < 6$, we consider only $y = 2.7861, 4.2461$. For $y = 2.7861, 4.2461$, by (9), we have $x = 1.3061, 7.3344$. Since $x < 6$, the solutions should be $x = 1.3061$ and $y = 2.7861$. By (4), we have $V = 149.333$, and this volume is bigger than 128, which is the volume of the cup in Problem 1.

Our next aim is to mathematically solve Equation (10), and we use the standard method for finding solutions to cubic equations. See [3] and [4].

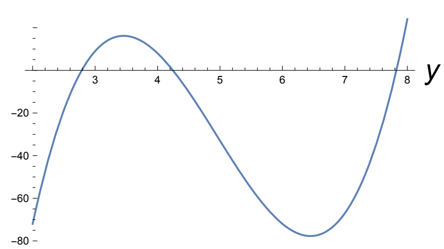

Figure 10: Graph of $7y^3 - 104y^2 + 468y - 648$

By (10),

$$y^3 - \frac{104y^2}{7} + \frac{468y}{7} - \frac{648}{7} = 0\,. \tag{11}$$

Let

$$y = z + \frac{104}{21}\,. \tag{12}$$

By (11) and (12) we have

$$z^3 - \frac{988}{147}z - \frac{40696}{9261} = 0. \tag{13}$$

Let $z = u + v$. Then, we have

$$u^3 + v^3 - \frac{40696}{9261} + 3\left(uv - \frac{988}{441}\right)(u+v) = 0. \tag{14}$$

Next, we look for u and v such that

$$u^3 + v^3 - \frac{40696}{9261} = 0 \qquad \text{and} \qquad uv - \frac{988}{441} = 0\,. \tag{15}$$

It is clear that you can get (14) from (15), but not (15) from (14). Readers who are not familiar with the method for solving cubic equations may wonder why we use (15) in the following. As you see later in (23), you get three solutions of (13). Then, you can understand why we use (15).

By (15),

$$u^3 + v^3 = \frac{40696}{9261} \qquad \text{and} \qquad u^3v^3 = \left(\frac{988}{441}\right)^3$$

and hence u^3 and v^3 are solutions of the equation

$$t^2 - \frac{40696}{9261}t + \left(\frac{988}{441}\right)^3\,. \tag{16}$$

Therefore, we have

$$u^3 = \frac{20348 + 2268\sqrt{107}\,i}{9261} \tag{17}$$

$$v^3 = \frac{20348 - 2268\sqrt{107}\,i}{9261}\,. \tag{18}$$

There exist real numbers r, θ such that $r > 0$, $0 < \theta < \pi/2$ and

$$r(\cos\theta + i\sin\theta) = \frac{20348 + 2268\sqrt{107}\,i}{9261}\,. \tag{19}$$

Then, we have

$$r = \frac{1976\sqrt{247}}{9261} \tag{20}$$

and

$$\theta = \tan^{-1}\left(\frac{567\sqrt{107}}{5087}\right)\,. \tag{21}$$

Therefore, by (17), (18) and (19) we have

$$u = \omega^i r^{1/3}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$$

for $i = 0, 1, 2$ and

$$v = \omega^j r^{1/3}\Big(\cos\frac{-\theta}{3} + i \sin\frac{-\theta}{3}\Big)$$

for $j = 0, 1, 2$, where $\omega = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$.

By (15), uv is a real number, and hence we have

$$\begin{aligned} u &= \omega^{3-k} r^{1/3}\Big(\cos\frac{\theta}{3} + i \sin\frac{\theta}{3}\Big) \\ v &= \omega^{k} r^{1/3}\Big(\cos\frac{-\theta}{3} + i \sin\frac{-\theta}{3}\Big) \end{aligned} \tag{22}$$

for $k = 0, 1, 2$. Then, the solutions of (13) are

$$\begin{aligned} z = u + v = r^{1/3}&\Bigg(\cos\left(\frac{\theta}{3} + \frac{2(3-k)\pi}{3}\right) + i \sin\left(\frac{\theta}{3} + \frac{2(3-k)\pi}{3}\right)\Bigg) \\ &+ r^{1/3}\Bigg(\cos\left(-\frac{\theta}{3} + \frac{2k\pi}{3}\right) + i \sin\left(-\frac{\theta}{3} + \frac{2k\pi}{3}\right)\Bigg) \end{aligned} \tag{23}$$

for $k = 0, 1, 2$. Here we have

$$\sin\left(\frac{\theta}{3} + \frac{2(3-k)\pi}{3}\right) + \sin\left(-\frac{\theta}{3} + \frac{2k\pi}{3}\right) = 0\,, \tag{24}$$

since

$$\left(\frac{\theta}{3} + \frac{2(3-k)\pi}{3}\right) + \left(-\frac{\theta}{3} + \frac{2k\pi}{3}\right) = 2\pi\,. \tag{25}$$

By (12), (23) and (24), we have

$$y = z + \frac{104}{21} = r^{1/3}\cos\left(\frac{\theta}{3} + \frac{2(3-k)\pi}{3}\right) + r^{1/3}\cos\left(-\frac{\theta}{3} + \frac{2k\pi}{3}\right) + \frac{104}{21}\,,$$

and hence, by (25),

$$y = 2r^{1/3}\cos\left(\frac{\theta}{3} + \frac{2k\pi}{3}\right) + \frac{104}{21} \tag{26}$$

for $k = 0, 1, 2$. Now we have three values for y, and we must choose the right one. Here we use approximate values for y. By (20), (21) and (26) we have

$$y = 7.82482\,,\ 2.78616\,,\ 4.24616 \tag{27}$$

for $k = 0, 1, 2$. Since $0 < x < y$, by (9) we have

$$0 < \frac{-5y^2 + 54y - 108}{y} < y\,.$$

Therefore, $0 \leq -5y^2 + 54y - 108 < y^2$ and hence

$$5y^2 - 54y + 108 < 0 \qquad \text{and} \qquad 6y^2 - 54y + 108 > 0\,. \tag{28}$$

By (28), we have

$$\frac{27-3\sqrt{21}}{5} < y < \frac{27+3\sqrt{21}}{5} \tag{29}$$

and

$$y < 3 \quad \text{or} \quad y > 6\,. \tag{30}$$

Since $y < 6$, (29) and (30) imply that

$$\frac{27-3\sqrt{21}}{5} < y < 3\,, \tag{31}$$

where $\frac{27-3\sqrt{21}}{5}$ is approximately equal to 2.65.

By (27), the only value of y that satisfies (31) is $y = 2.78616$; that is when $k = 1$. Therefore by (26), the value of y that gives the maximal volume is

$$y = 2r^{1/3}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \frac{104}{21}\,. \tag{32}$$

By (9) and (32), we also get the exact value of y that gives the maximal volume of V.

3 Appendix

In mathematics research, mathematics software is often beneficial. The authors used a desktop version of Mathematica, but online Mathematica is free with some restrictions on the time used. You can also use Wolfram Alpha for free, but Wolfram Alpha has a professional version that is more useful and not free.

Here the authors present some Mathematica programming.

Example 1. *This is a Mathematica program to calculate the maximum volume of the square frustum in Problem 1.*

```
data=Table[4/3(x^2+x*y-18x-18y+y^2+108)Sqrt[y^2-(y-x)^2],
{y,0,6,0.01},{x,0,y,0.01}];Max[data]
```

The output is 149.333. *This is a Mathematica program to find x and y that gives the maximal volume of V.*

```
MaximalBy[Flatten[Table[{x,y,4/3(x^2+x*y-18x-18y+y^2+108)
Sqrt[y^2-(y-x)^2]},{y,0,6,0.01},{x,0,y,0.01}],1],Last]
```

The output is $\{\{1.31, 2.79, 149.333\}\}$.

Example 2. *This is a Mathematica program to find numerical solutions to $7y^3 - 104y^2 + 468y - 648 = 0$.*

```
Nsolve[7y^3-104 y^2+468 y-648 == 0,y]
```

The output is as follows.

```
{{y -> 2.78616}, {y -> 4.24616}, {y -> 7.82482}}
```

Example 3. *This is a Mathematica program to get the 3D graph of the volume in Problem 2.*

```
Plot3D[4/3(x^2+x*y-18x-18y+y^2+108)Sqrt[y^2-(y-x)^2],
{y,0,6},{x,0,y},AxesLabel -> Automatic]
```

The output is as following.

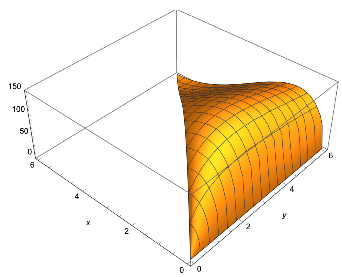

Figure 11

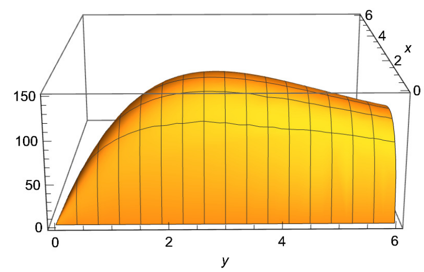

Figure 12

4 Some comments on this research project

As you see in the preceding sections, you can start research if you change the conditions of a problem in a standard textbook. Sometimes, the problem becomes complicated and demands more knowledge, but you can use mathematical software to supplement your knowledge. Usually, the textbook is for you to understand the problems, but if you are willing to change some parts of the problems, then you can begin to experience the enormous joy of mathematical research.

Acknowledgements

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References

[1] R. Miyadera, D. Ikeda, W. Ogasa, T. Inoue, T. Nishimura and T. Nakaoka, The maximization of a cup made from a square sheet of paper, *Science of Origami* **1(1)** (2011) (in Japanese).

[2] E. Weisstein, Pyramidal Frustum, *MathWorld–A Wolfram Web Resource*, https://mathworld.wolfram.com/PyramidalFrustum.html, last access on 2023-07-12.

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