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Problems 1711–1720

We have recently been looking through problems which have been posed and solved in the nearly 60–year lifespan of *Parabola*. Doing so suggested ideas for a number of new problems, some of which we publish this issue. We have given references to the previous problems, and if you wish, you can look up the solutions for a hint to the present problems.

Parabola would like to thank Toyesh Prakash Sharma for contributing Problem 1718.

Q1711 The numbers $1, 2, \ldots, 64$ are written onto the squares of an 8×8 chessboard, one to a square. In Problem 640[1], it was proved that there must be two squares sharing a side or a corner which contain numbers differing by 16 or less.

(a) Find such an arrangement in which the minimum difference between numbers in squares which are adjacent horizontally, vertically or diagonally is 15.

(b) Prove that there is no such arrangement in which the minimum difference is greater than 15.

Q1712 Let $f(x)$ be a cubic polynomial with leading coefficient 4, say,

$$f(x) = 4x^3 + a_2x^2 + a_1x + a_0\,,$$

with the property that $|f(x)| \leq 1$ whenever $|x| \leq 1$. Prove that the only possible such polynomial is $f(x) = 4x^3 - 3x$. (We showed in Problem 523[2] that no cubic with leading coefficient greater than 4 has the stated property.)

Q1713 In Problem 1240[3], we proved the following fact about the set $S = \{\,0, 1, 2\,\}$: if $f(x) = ax^2 + bx + c$ is any quadratic polynomial with real coefficients such that $f(x)$ is an integer for all values of x in S, then $f(x)$ is an integer for all integer values of x.

(a) Find all possible sets of three integers which have the same property.

(b) Can you find a set of four or more integers which does not include S or any of the other sets you found in (a), and which still has the same property?

Q1714 In Problem 1089[4], we described an apartment block of 120 apartments. Every day, the inhabitants of an "impossible" apartment – one having 15 or more residents – all go off to other apartments in the same block, no two to the same apartment. We showed that if there are 119 residents, then sooner or later there will be no more impossible apartments. Now suppose that one more person moves in, so that there are 120 residents. Devise a scenario in which there will always be an impossible apartment.

[1] *Parabola* **21(2)** (1985).
[2] *Parabola* **18(1)** (1982).
[3] *Parabola* **43(1)** (2007).
[4] *Parabola* **36(3)** (2000).

Q1715 In Problem 666[5], we showed that if A is any finite, non–empty subset of

$$S = \{\, 2,\, 2^3,\, 2^5, \dots \}\,,$$

then the sum of the elements of A cannot be a perfect square; and we also did something similar for cubes.

Now find an infinite set S of positive integers such that for any finite non–empty subset A of S, the sum of all elements of A is never a perfect power. By a *perfect power*, we mean a positive integer a^b, where a, b are positive integers and $b > 1$.

Q1716 A bank safe has a security keypad consisting of buttons which bear the digits 0 to 9; a password to gain access to the safe consists of six digits (with repeated digits being permitted). Because of the keypad's innovative design, a valid password must be such that the maximum difference between any two of its digits is exactly equal to n, a number from 1 to 9; it is up to the bank's security office to decide what value of n should be specified. Eventually they decide that in order to provide the maximum different number of passwords, they will just go with $n = 9$. Is this a wise choice?

Q1717 Find the smallest perfect square whose decimal representation consists of the same block of digits twice over. (An example of such a number would be 123123 – but of course, that's not a square.) As usual, the first digit of a number may not be zero.

Q1718 Let a, b, c be positive numbers. Prove that if $\sqrt{a} + \sqrt{b} + \sqrt{c} \le 4$, then

$$\frac{a}{7+a^4} + \frac{b}{7+b^4} + \frac{c}{7+c^4} \le \frac{1}{2}\,.$$

Q1719 Find all integers $n > 0$ for which $n^2 + 2023$ and $(n+1)^2 + 2023$ have a common factor greater than 1.

Q1720 Prove that if $a_1, a_2, \dots, a_n$ are positive numbers, and M is a positive constant such that

$$a_1 < M^2\,, \quad a_2 < M^4\,, \quad \dots, \quad a_n < M^{2^n}\,,$$

then

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots + \sqrt{a_n}}}} < M\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}\,,$$

where there are n square root symbols on the right–hand side.

[5] *Parabola* **22(1)** (1984).