Parabola Volume 59, Issue 3 (2023)

On the general formula of Alcuin's sequence: A combinatorial and algorithmic approach

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1 Introduction

Let $T(P)$ be the number of non-congruent triangles with perimeter P and integral side lengths. Alcuin's Sequence is the infinite sequence $T(3), T(4), T(5), \ldots$. In Ross Honsberger's book *Mathematical Gems, Vol. 3* [1], the author acknowledges a solution using partitions, by George Andrews, to the problem of determining $T(P)$. Andrew's solution yields the following formula, in which $[x]$ is the nearest integer to x.

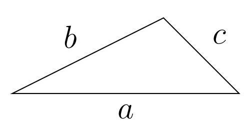
Theorem 1.

$$T(P) = \begin{cases} \left[\dfrac{P^2}{48}\right] & \text{, if } P \text{ is even;} \\ \left[\dfrac{(P+3)^2}{48}\right] & \text{, if } P \text{ is odd.} \end{cases}$$

In this paper, I will present an algorithm and an elementary derivation of this formula for $T(P)$ without the use of partitions. For computational verification and graphing purposes, the Python Programming Language and its Matplotlib library will be used.

2 Two conditions

Given a natural number $P \geq 3$, we would like to find $T(P)$, the number of non-congruent triangles with perimeter P and integral side lengths.



The condition on a, b, c is

$$a + b + c = P\,. \tag{1}$$

We also observe the inequalities

$$\begin{aligned} a + b &> c \\ a + c &> b \\ b + c &> a\,. \end{aligned} \tag{2}$$

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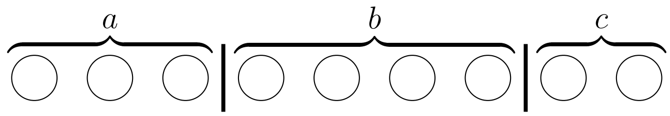
We shall first find out how many ordered triplets (a, b, c) satisfy condition (1). Then, we will use our results to find the number of unordered triplets $\{a, b, c\}$ satisfying this condition. Afterwards, we shall find out how many of the unordered triplets $\{a, b, c\}$ that follow condition (1) also satisfy the inequalities (2).

Finding the number of *ordered* triplets satisfying condition (1)

Consider a row of P balls, and place two dividing lines in the spaces between the balls, representing the two $+$ signs of the equation $a + b + c = P$:



The lines divide the P balls into three parts containing a, b and c balls, respectively. In the figure above, $a = 3$, $b = 4$ and $c = 2$. By placing the two dividers in all possible ways between the balls, we will get all ordered positive-integer triples (a, b, c) whose sum is P. The number of these triples is therefore the same as the number of ways in which to choose the positions of the two dividing lines. There are $P-1$ spaces between the balls, and the order of the dividing lines does not matter. The number of ways in which to choose the two positions for the lines is therefore

$$\binom{P-1}{2} = \frac{(P-1)!}{2!\big((P-1)-2\big)!} = \frac{(P-1)(P-2)}{2}\,.$$

This number is called `total_ordered` in the generalised algorithm (see Section 4). We have found the total number of *ordered* triplets (a, b, c) satisfying condition (1). Now, we will find the number of *unordered* triplets $\{a, b, c\}$ satisfying this condition.

Finding the number of *unordered* triplets satisfying condition (1)

The unordered triplets $\{a, b, c\}$ must fall into one and only one of the following cases:

Case 1 a, b, c are distinct.

Case 2 Exactly two of the numbers a, b, c are identical.

Case 3 a, b, c are all the same number.

Let us start with the third case. If P is divisible by 3, then there is exactly one triple $\{a, b, c, \}$, namely the one in which $a = b = c = \frac{P}{3}$. Otherwise, there is no triple. This number - 1 or 0 - is called `case_3` in the generalised algorithm.

Now consider the second case. Two of the numbers a, b, c are the same; we denote them by x and denote the third number by y. Then $2x + y = P$, so $x = \frac{P-y}{2}$. Since $y \geq 1$,

$$x \leq \frac{P-1}{2}\,.$$

Since this fraction might not be a natural number, we can round it down to its nearest integer with the *floor function* and thereby get a precise upper bound:

$$x \leq \left\lfloor \frac{P-1}{2} \right\rfloor .$$

The unordered triplets $\{a, b, c\}$ for the second case therefore range from $\{1, 1, P-2\}$ to $\{x_{max}, x_{max}, 1 \text{ or } 2\}$, where $x_{max} = \lfloor \frac{P-1}{2} \rfloor$ is the number of possible unordered triplets.

Note that these triples include any triple counted in Case 3. Since Case 2 only includes triples with exactly two identical elements, the number of unordered triples in Case 2, expressed as `case_2` in the generalised algorithm, is

$$\texttt{case\_2} = x_{max} - \texttt{case\_3} = \left\lfloor \frac{P-1}{2} \right\rfloor - \texttt{case\_3} .$$

Let us now look at Case 1. We will find the number of *ordered* triplets corresponding to Case 2 and 3 and subtract their sum from `total_unordered`. This will give us the number of *ordered* triplets for Case 1, from which we will find the number of *unordered* triples for Case 1.

The *ordered* triplets corresponding to Case 2 and 3 are

$$(x, x, y)\,,\ (x, y, x)\,,\ (y, x, x) \qquad \text{and} \qquad (x, x, x)$$

so the number of *ordered* triplets corresponding to these cases is

$$3 \times \texttt{case\_2} \quad \text{and} \quad 1 \times \texttt{case\_3} = \texttt{case\_3},$$

respectively. Hence, the number of *ordered* triplets corresponding to Case 1 is

$$\texttt{total\_ordered} - (3 \times \texttt{case\_2} + \texttt{case\_3}) .$$

We divide this number by 6 to find the number of *unordered* triples in Case 1, since each unordered triple $\{a, b, c\}$ of distinct numbers can be ordered in $3! = 6$ different ways:

$$(a, b, c)\,,\ (a, c, b)\,,\ (b, a, c)\,,\ (b, c, a)\,,\ (c, a, b)\,,\ (c, b, a) .$$

The number of *unordered* triples in Case 1 is therefore

$$\texttt{case\_1} = \frac{1}{6}\big(\texttt{total\_ordered} - 3 \times \texttt{case\_2} - \texttt{case\_3}\big) .$$

Unordered triplets satisfying condition (1) but not inequalities (2)

When looking at the unordered triples, we can assume that the side lengths satisfy $a \le b \le c$. Then

$$\begin{aligned} a + c &> b \\ b + c &> a\,. \end{aligned}$$

If condition (2) is not satisfied, then the remaining inequality $a + b > c$ cannot be true. Let us find the number of unordered triplets for which $a + b \le c$.

First note that $c \ge a + b \ge 2$ and that $c = P - a - b \le P - 2$. Also, note that $c \ge a + b = P - c$, so $c \ge \frac{P}{2}$. Since c is an integer, $c \ge \left\lceil \frac{P}{2} \right\rceil$ where $\lceil x \rceil$ is the *ceiling* function that rounds any real number x up to its nearest integer.

Suppose that $c \ge \left\lceil \frac{P}{2} \right\rceil$. To find the number of unordered pairs $\{a, b\}$ that satisfy $1 \le a, b \le c$ and $a + b = P - c$, we could start by setting $a = 1$ and $b = P - c - 1$. Then we can increase a by 1 and decrease b by 1 to get another pair, and continue until a and b meet, when $a = b = \frac{P-c}{2}$, or when $a = b - 1 = \frac{P-c-1}{2}$. The number of unordered pairs $\{a, b\}$ is then $\frac{P-c}{2}$ when $P - c$ is even and $\frac{P-c-1}{2}$ when $P - c$ is odd. We can write these numbers compactly with the one expression

$$\left\lfloor \frac{P-c}{2} \right\rfloor .$$

To find the number of unordered triplets $\{a, b, c\}$ satisfying condition (1) but not condition (2), we have to sum over $\left\lfloor \frac{P-c}{2} \right\rfloor$, where c ranges from $\left\lceil \frac{P}{2} \right\rceil$ to $P - 2$, or in other words, $P - c$ ranges from 2 to $P - \left\lceil \frac{P}{2} \right\rceil = \left\lfloor \frac{P}{2} \right\rfloor$:

$$\sum_{i=2}^{\left\lfloor \frac{P}{2} \right\rfloor} \left\lfloor \frac{i}{2} \right\rfloor . \tag{3}$$

This sum is essentially `complementary_condition` in the generalised algorithm. This value is subtracted from `case_1` + `case_2` + `case_3` to get the number of unordered triples $\{a, b, c\}$ satisfying conditions (1) and (2).

3 An explicit formula for *T(P)*

To express the sum (3) by the explicit expressions such as those in Theorem 1, we use the formula

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2} n(n+1)\,. \tag{4}$$

In (3), we observe that the summand $\left\lfloor \frac{i}{2} \right\rfloor$ is $\frac{i}{2}$ when i is even, and is $\frac{i-1}{2}$ when i is odd.

Suppose that the upper bound of (3), $Q = \left\lfloor \frac{P}{2} \right\rfloor$, is even. Then

$$\sum_{i=2}^{Q} \left\lfloor \frac{i}{2} \right\rfloor = \sum_{i=2,\ 2|i}^{Q} \frac{i}{2} + \sum_{i=3,\ 2\nmid i}^{Q-1} \frac{i-1}{2} = \sum_{i=2,\ 2|i}^{Q} \frac{i}{2} + \sum_{i=2,\ 2|i}^{Q-2} \frac{i}{2} = \sum_{j=1}^{Q/2} j + \sum_{j=1}^{(Q-2)/2} j = \frac{Q}{2} + 2 \sum_{j=1}^{(Q-2)/2} j\,.$$

Now we apply the formula (4) to get

$$\sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor = \frac{Q}{2} + \frac{Q-2}{2}\Big(\frac{Q-2}{2}+1\Big) = \frac{Q^2}{4}\,.$$

Next, suppose that $Q = \lfloor \frac{P}{2} \rfloor$ is odd:

$$\sum_{i=2}^{Q} \left\lfloor \frac{i}{2} \right\rfloor = \sum_{i=2,\, 2|i}^{Q-1} \frac{i}{2} + \sum_{i=3,\, 2\nmid i}^{Q} \frac{i-1}{2} = \sum_{i=2,\, 2|i}^{Q-1} \frac{i}{2} + \sum_{i=2,\, 2|i}^{Q-1} \frac{i}{2} = \sum_{j=1}^{(Q-1)/2} j + \sum_{j=1}^{(Q-1)/2} j = 2\sum_{j=1}^{(Q-1)/2} j\,.$$

By applying the formula (4), we get

$$\sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor = \frac{Q-1}{2}\Big(\frac{Q-1}{2}+1\Big) = \frac{Q^2-1}{4}\,. \tag{5}$$

We will now find an explicit expression for the number of unordered triples $\{a, b, c\}$ that satisfy conditions (1) and (2). First note that

$$\begin{aligned}
T(P) &= \texttt{case_1} + \texttt{case_2} + \texttt{case_3} - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \frac{1}{6}\big(\texttt{total_ordered} - 3 \times \texttt{case_2} - \texttt{case_3}\big) + \texttt{case_2} + \texttt{case_3} - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \frac{1}{6}\big(\texttt{total_ordered} + 3 \times \texttt{case_2} + 5 \times \texttt{case_3}\big) - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \frac{1}{6}\Big(\texttt{total_ordered} + 3\Big(\Big\lfloor \frac{P-1}{2} \Big\rfloor - \texttt{case_3}\Big) + 5 \times \texttt{case_3}\Big) - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \frac{1}{6}\Big(\texttt{total_ordered} + 3\left\lfloor \frac{P-1}{2} \right\rfloor + 2 \times \texttt{case_3}\Big) - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor \\
&= \frac{1}{6}\Big(\frac{(P-1)(P-2)}{2} + 3\left\lfloor \frac{P-1}{2} \right\rfloor + 2 \times \texttt{case_3}\Big) - \sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor\,.
\end{aligned}$$

The term $\lfloor \frac{P-1}{2} \rfloor$ depends on whether P is even or odd; the term `case_3` depends on whether P is divisible by 3; and the sum depends on whether $Q = \lfloor \frac{P}{2} \rfloor$ is even or odd, which in turn depends on which remainders P has when divided by 4. This gives twelve different cases to consider, each of which we can express by the remainder r when dividing P by 12; that is, $P = 12k + r$ where k is some integer and $r \in \{0, 1, \ldots, 11\}$. For each value of r, we can calculate each term of the expression for $T(P)$ above to get the expression for $T(P)$. For instance, for $r = 5$, $\texttt{case\_3} = 0$

since $P = 12k + 5$ is not divisible by 3. Also, $\lfloor \frac{P-1}{2} \rfloor = \lfloor \frac{12k+4}{2} \rfloor = 6k + 2 = \frac{P-1}{2}$, and $Q = \lfloor \frac{P}{2} \rfloor = \frac{P-1}{2} = 6k + 2$ is even, so

$$\sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor = \frac{Q^2}{4} = \frac{(P-1)^2}{16} .$$

Therefore,

$$T(P) = \frac{1}{6}\left(\frac{(P-1)(P-2)}{2} + \frac{3(P-1)}{2} + 2 \times 0 \right) - \frac{(P-1)^2}{16} = \frac{P^2 + 6P - 7}{48} .$$

The calculated terms for all $r = 0, 1, \ldots, 11$ are given in the table below.

r	$\left\lfloor \frac{P-1}{2} \right\rfloor$	`case_3`	$Q = \lfloor \frac{P}{2} \rfloor$	Q Even/Odd	$\sum_{i=2}^{\lfloor \frac{P}{2} \rfloor} \left\lfloor \frac{i}{2} \right\rfloor$	$T(P)$
0	$\frac{P-2}{2}$	1	$\frac{P}{2}$	Even	$\frac{P^2}{16}$	$\frac{P^2}{48}$
1	$\frac{P-1}{2}$	0	$\frac{P-1}{2}$	Even	$\frac{(P-1)^2}{16}$	$\frac{(P+3)^2-16}{48}$
2	$\frac{P-2}{2}$	0	$\frac{P}{2}$	Odd	$\frac{P^2-4}{16}$	$\frac{P^2-4}{48}$
3	$\frac{P-1}{2}$	1	$\frac{P-1}{2}$	Odd	$\frac{(P-1)^2-4}{16}$	$\frac{(P+3)^2+12}{48}$
4	$\frac{P-2}{2}$	0	$\frac{P}{2}$	Even	$\frac{P^2}{16}$	$\frac{P^2-16}{48}$
5	$\frac{P-1}{2}$	0	$\frac{P-1}{2}$	Even	$\frac{(P-1)^2}{16}$	$\frac{(P+3)^2-16}{48}$
6	$\frac{P-2}{2}$	1	$\frac{P}{2}$	Odd	$\frac{P^2-4}{16}$	$\frac{P^2+12}{48}$
7	$\frac{P-1}{2}$	0	$\frac{P-1}{2}$	Odd	$\frac{(P-1)^2-4}{16}$	$\frac{(P+3)^2-4}{48}$
8	$\frac{P-2}{2}$	0	$\frac{P}{2}$	Even	$\frac{P^2}{16}$	$\frac{P^2-16}{48}$
9	$\frac{P-1}{2}$	1	$\frac{P-1}{2}$	Even	$\frac{(P-1)^2}{16}$	$\frac{(P+3)^2}{48}$
10	$\frac{P-2}{2}$	0	$\frac{P}{2}$	Odd	$\frac{P^2-4}{16}$	$\frac{P^2-4}{48}$
11	$\frac{P-1}{2}$	0	$\frac{P-1}{2}$	Odd	$\frac{(P-1)^2-4}{16}$	$\frac{(P+3)^2-4}{48}$

The parity of P depends on the parity of r: if r is even, then $P = 12k + r$ is also even, and vice versa. From the table, we see that, when P is even,

$$T(P) = \frac{P^2 + m}{48}$$

where $m \in \{0, -4, -16, 12\}$. For each of these numbers m, $|m|$ is less than $\frac{48}{2}$. Hence, for the case when P is even, we can express $T(P)$ more compactly as:

$$T(P) = \left[\frac{P^2}{48} \right]$$

where $[x]$ rounds a number x up or down to its nearest integer.

Similarly, for the case when P is odd,

$$T(P) = \frac{(P+3)^2 + m}{48}$$

where $m \in \{-16, 12, -4, 0\}$, so

$$T(P) = \left[\frac{(P+3)^2}{48}\right]$$

This proves Theorem 1. □

4 The generalised algorithm

In the previous section, we derived the algebraic expressions for $T(P)$ in Theorem 1 from the observations made in Section 2. This involved lengthy calculations and case-by-case considerations. We could have saved some of this effort by using the following generalised algorithm for calculating $T(P)$ for any given perimeter P. The algorithm is presented as a function in Python Programming Language below.

```
def T(P):
    total_ordered = ((P-1)*(P-2))/2
    if P % 3 == 0:
        case_3 = 1
    else:
        case_3 = 0

    case_2 = int((P-1)/2) - case_3

    case_1 = (total_ordered - (3*case_2) - case_3)/6
    total_unordered = case_1 + case_2 + case_3

    Q = int(P/2)

    if Q % 2 == 0:
        complementary_condition = (Q**2)/4
    else:
        complementary_condition = ((Q**2)-1)/4

    result = total_unordered - complementary_condition
    return result
```

Computational Verification

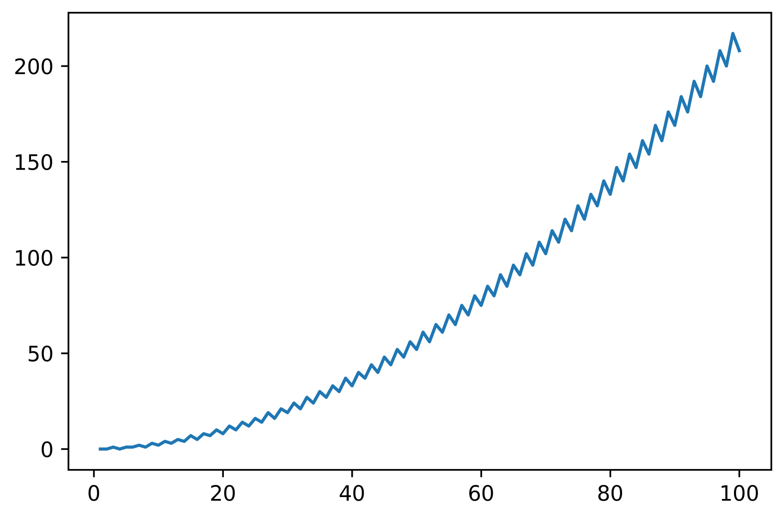
We will now computationally verify our algorithm by defining another function that computes $T(P)$ for a given perimeter P by going through each possible case. The output of our generalised algorithm is then compared with the results of the brute force algorithm. The brute force algorithm is written in the Python Programming Language as follows:

```
def triangle_count(P):
    count = 0
    li_1 = []
    for a in range(1, P+1):
        for b in range(1, P+1):
            for c in range(1, P+1):
                li_2 = [a, b, c]
                li_2 = sorted(li_2)
                if a+b+c == P:
                    if a+b > c and a+c > b and b+c > a:
                        if li_2 not in li_1:
                            count += 1
                            li_1.append(li_2)
    return count
```

Using the following code, we can check whether or not our generalised algorithm is correct and plot a graph of $T(P)$ as a function of P. We need to check our results within a certain range of P and choose the range $P = 1, \ldots, 100$.

```
import matplotlib.pyplot as plt

start = 1
end = 100
error = 0

li_P = []
li_T_P = []

for P in range(start, end+1):
    li_P.append(P)
    li_T_P.append(T(P))
    if T(P) != triangle_count(P):
        error += 1

print(error)
plt.figure(dpi = 1200)
plt.plot(li_P, li_T_P)
plt.savefig("Figure_3.png")
```

Running the program, we find that `error` $= 0$, which means that our algorithm is correct within the range. We also get the following plot of $T(P)$ for $P = 1, \ldots, 100$:



This graph is shown as a continuous graph but the domain of the function $T(P)$ is only the natural numbers. The data points were connected with straight lines only to illustrate the trend between two adjacent values of $T(P)$.

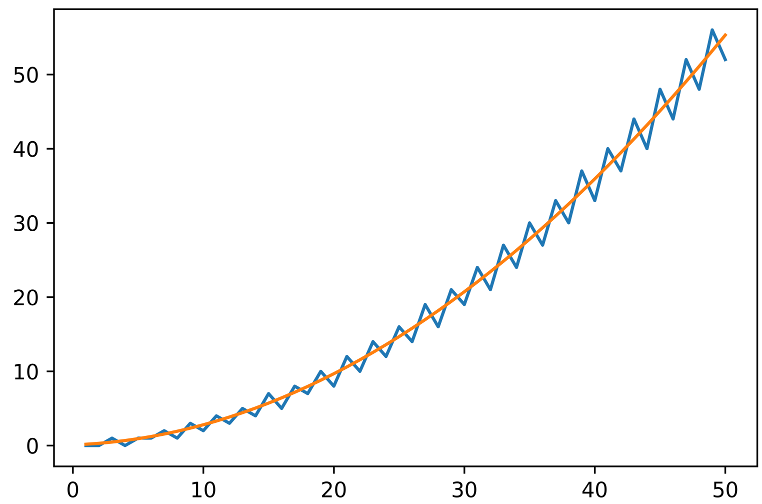
From the jig-jagged structure of the graph, $T(P)$ does not seem like a simple polynomial function. This is because the algorithm runs differently for even and odd numbers. This is reflected in the piece-wise formula in Theorem 1.

5 Further experimentation

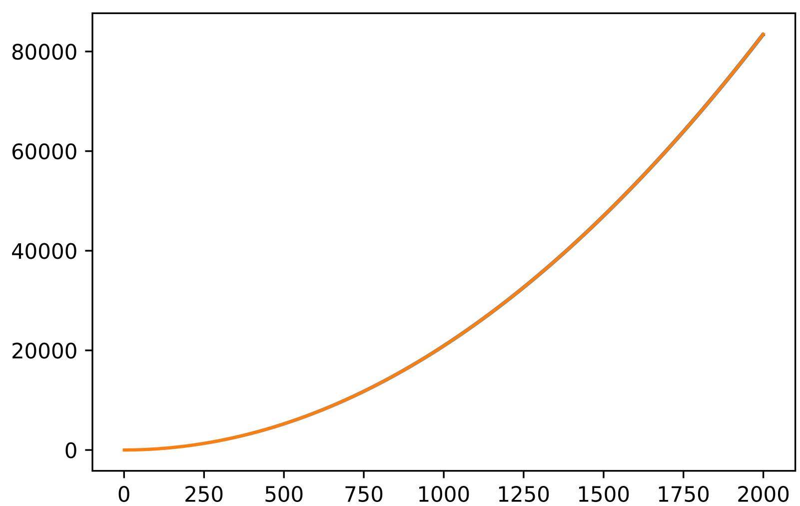
We can also predict the best-fit polynomial equation representing the long-term trend of $T(P)$. To do this, we can find the arithmetic mean of the even and odd expressions for $T(P)$. We can ignore the rounding of the expressions since they will have negligible effect on $T(P)$ for large values of P. We find the value approximating $T(P)$ for large values of P to be

$$T_{\text{avg}}(P) = \frac{P^2 + 3P + 9/2}{48} \approx \frac{P(P+3)}{48}\,.$$

Plotting $T_{\text{avg}}(P)$ and $T(P)$ simultaneously for $P = 1, \ldots, 50$ gives the following graph:



Plotting $T(P)$ and $T_{\text{avg}}(P)$ simultaneously for a large range, say $P = 1, \ldots, 2000$, shows the large scale behaviour of $T(P)$:



At this scale, it appears as if both graphs coincide.

6 Conclusion

We have explored the problem of finding the number $T(P)$ of distinct triangles with a given perimeter P and integral side-lengths. Through a combinatorial approach, we have established a formula and an algorithm for calculating $T(P)$. We see that the overall behavior of $T(P)$ is quadratic.

Acknowledgement

I would like to express my gratitude to the Editor in Chief of *Parabola*, Dr. Thomas Britz, for his constant support throughout the revisions of the paper. I am grateful for his suggestions and editing for better presentation and delivery of ideas.

References

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