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Applications of the Pigeonhole Principle in mathematics

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1 Introduction

We will explore *The Pigeonhole Principle*, a fundamental theorem in mathematics. Simply expressed, if you were to place at least $n+1$ items ("pigeons") in only n boxes ("pigeonholes"), then at least one box will contain at least two items.

You might think that this is an obvious fact - when you try to fit 6 apples in 4 lunchboxes, it's not possible to have only one apple in each lunchbox - and you would be correct! This paper will explore how such a seemingly simple theorem has important applications in more difficult mathematics, ranging from geometry to number theory and algebra.

2 The Pigeonhole Principle

Let us state the Principle more formally:

Theorem 1 (The Pigeonhole Principle)**.** *If more than n pigeons are placed in n pigeonholes, then at least one pigeonhole will contain at least two pigeons.*

Proof. Another way to express this Principle is as follows: if no pigeonhole has least two pigeons - that is, each pigeonhole contains at most one pigeon - then it cannot be true that more than n pigeons were placed in these n pigeonholes. That is naturally a true statement: if each of the n pigeonholes contains at most one pigeon, then there can be at most n pigeons in the n pigeonholes. □

Now, if you try to place 9 apples into 4 lunch pigeonholes, then by the Pigeonhole Principle, some lunch pigeonhole will contain at least two apples. What's even more *cool* is that some lunch pigeonhole will also contain at least three apples! Why?

Theorem 2 (The (General) Pigeonhole Principle)**.** *If more than mn pigeons are placed in n pigeonholes, then at least one pigeonhole will contain more than m pigeons.*

Proof. If each of the n pigeonholes contains at most m pigeons, then there can be at most mn pigeons. This simple observation is another way to express the theorem. □

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3 Applications to problem solving

Now let's see how the Pigeonhole Principle can be used to solve problems.

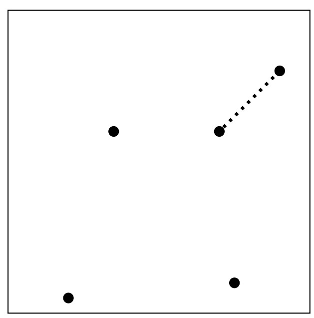
Example 3. *Prove that, if we pick* 6 *different numbers from* $\{1, 2, \ldots, 10\}$, *then we can choose two of them such that they add up to* 11.

Proof. You may ask: We are picking numbers, but how does the Pigeonhole Principle apply here?

Let 5 "pigeonholes" be the sets $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$, respectively. If we pick 6 different numbers - "pigeons" - from $\{1, 2, \ldots, 10\}$, then there are more pigeons than pigeonholes, so, by the Pigeonhole Principle, some pigeonhole must contain at least two pigeons. That is, at least one the five sets contains two of the picked numbers. Whichever set this is, we have two numbers that add up to 11! $\square$

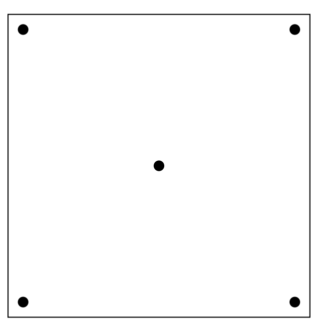
Let's now look at a geometry example.

Example 4. *Given five points on the interior of a square with side length* 2, *prove that two of the points at distance less than* 1.5 *apart.*



Bonus question: What is the smallest real constant you can replace 1.5 *with so that the statement is still true?*

Proof. The crux of this problem lies in how to set up the Pigeonhole Principle. What are the pigeons, and what are the pigeonholes? After trying to keep the points as far as possible from each other, we notice that this intuitively occurs when four points lie in the four corners and the fifth lies in the middle of the square:

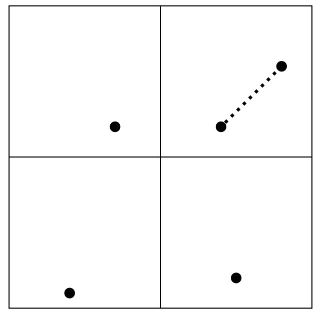


Here, the distances are just under the diagonal distance of a unit square, namely

$$\sqrt{2} \approx 1.414$$

which is less than 1.5! Let us prove this rigorously.

Divide the 2×2 square into four 1×1 squares:



Since there are five points and only four 1×1 squares, at least one of these squares will contain two points. These two points have distance at most $\sqrt{2} < 1.5$. $\square$

4 Using the Pigeonhole Principle to obtain information

We now move onto a more complex application of the Pigeonhole Principle.

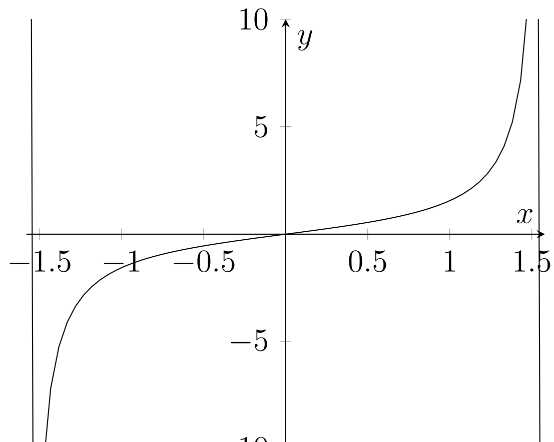
Example 5 (Based on 2021 Bored of Studies Mathematics Extension 2 Exam)**.**
Given any 7 real numbers, prove that at least two of them, x, y, satisfy

$$\left|\frac{x-y}{1+xy}\right| < \frac{1}{\sqrt{3}}. \tag{1}$$

Proof. While this problem seems difficult to approach at first, recall the tangent compound angle formula:

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{2}$$

Also, recall that each real number x can be written as $\tan A$ for some $A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



Now let our 7 real numbers be $\tan\alpha_1, \tan\alpha_2, \dots, \tan\alpha_7$ where $\alpha_1, \alpha_2, \dots, \alpha_7 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We need to show that two of these seven numbers, say $x = \tan\alpha_i$ and $y = \tan\alpha_j$, satisfy the inequality (1). By the identity (2), the inequality (1) can be expressed as

$$\left|\tan(\alpha_i - \alpha_j)\right| < \frac{1}{\sqrt{3}}.$$

Since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and $\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$, and $\tan x$ is a strictly increasing function when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can further re-express (1) as

$$-\frac{\pi}{6} < \alpha_i - \alpha_j < \frac{\pi}{6}$$

or, more simply,

$$\left|\alpha_i - \alpha_j\right| < \frac{\pi}{6}$$

This gives us an idea of how to apply the Pigeonhole Principle: There are 7 numbers in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which has length π, and we need the difference of two numbers to be less than $\frac{\pi}{6}$, exactly $\frac{1}{6}$ of the interval length. If we choose the "pigeonholes" to be the six intervals $\left(-\frac{3\pi}{6}, -\frac{2\pi}{6}\right], \left(-\frac{2\pi}{6}, -\frac{\pi}{6}\right], \dots, \left(\frac{2\pi}{6}, \frac{3\pi}{6}\right]$, then by the Pigeonhole Principle, one of the pigeonholes will have two of the numbers, α_i, α_j. Thus, $|\alpha_i - \alpha_j| < \frac{\pi}{6}$, so

$$\left|\frac{\tan\alpha_j - \tan\alpha_i}{1 + \tan\alpha_i \tan\alpha_j}\right| = \left|\tan(\alpha_j - \alpha_i)\right| < \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}},$$

as needed. □

We can see from this example that, while the Pigeonhole Principle might not have obvious use in a problem when you see it for the first time, it often appears after we perform simplifications. We can in fact generalise the above problem, as follows.

Example 6. *For each positive integer n, find the smallest positive real constant C_n such that, for any set of $n+1$ distinct real numbers $\{x_1, x_2, \dots, x_{n+1}\}$, there are at least two of them, say $x = x_i$ and $y = x_j$ with $i \neq j$, such that*

$$\left|\frac{x - y}{1 + xy}\right| < C_n.$$

Proof. Write $x_i = \tan\alpha_i$ where $\alpha_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for $i = 1, 2, \dots, n+1$. The n intervals $\left[-\frac{\pi}{2} + \frac{(k-1)\pi}{n}, -\frac{\pi}{2} + \frac{k\pi}{n}\right)$ for $k = 1, 2, \dots, n$ partition the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By the Pigeonhole Principle, at least one of the intervals must contain at least two of the $n+1$ numbers $\alpha_1, \alpha_2, \dots, \alpha_n$, say α_i and α_j. Then $|\alpha_i - \alpha_j| < \frac{\pi}{n}$, so

$$\left|\frac{\tan\alpha_j - \tan\alpha_i}{1 + \tan\alpha_i \tan\alpha_j}\right| = |\tan(\alpha_j - \alpha_i)| < \tan\frac{\pi}{n}.$$

Therefore, $C_n = \tan\frac{\pi}{n}$ always works! We will show that no smaller C_n works.

Assume that some $C_n < \tan\frac{\pi}{n}$ works, and write $C_n = \tan\left(\frac{\pi}{n} - \varepsilon\right)$ where $0 < \varepsilon < \frac{\pi}{n}$. For each $i = 1, 2, \ldots, n+1$, define $x_i = \tan\alpha_i$ where

$$\alpha_1 = -\frac{\pi}{2} + \varepsilon\,, \qquad \alpha_i = -\frac{\pi}{2} + \frac{(k-1)\pi}{n} \ \text{ for } \ i = 2, \ldots, n\,, \ \text{ and } \qquad \alpha_{n+1} = \frac{\pi}{2} - \varepsilon\,.$$

By assumption, we can find distinct $x = x_i$ and $y = x_j$ such that

$$\left|\frac{\tan\alpha_j - \tan\alpha_i}{1 + \tan\alpha_i \tan\alpha_j}\right| < C_n\,.$$

Thus,

$$\tan|\alpha_i - \alpha_j| < C_n = \tan\left(\frac{\pi}{n} - \varepsilon\right)$$

so

$$|\alpha_i - \alpha_j| < \frac{\pi}{n} - \varepsilon\,.$$

But, by definition, the distance between α_i and α_j is at least $\frac{\pi}{n} - \varepsilon$ or $\frac{\pi}{n}$, both of which are greater than $\frac{\pi}{n} - \varepsilon$, a contradiction! Therefore, our assumption that some $C_n < \tan\frac{\pi}{n}$ works, so the minimum possible C_n is $\tan\frac{\pi}{n}$. $\square$

Example 7 (Serbian Mathematical Olympiad 2016, by Dušan Djukić)**.**
Suppose $a_1, a_2, \ldots, a_{2^{2016}}$ are positive integers such that, for all n with $1 \le n \le 2^{2016}$,

$$a_n \le 2016 \quad \textit{and} \quad a_1 a_2 \cdots a_n + 1 \ \textit{ is a perfect square.}$$

Prove that at least one of the numbers $a_1, a_2, \ldots, a_{2^{2016}}$ must be equal to 1.

Proof. For each $n = 1, 2, \ldots 2^{2016}$, write $a_1 a_2 \cdots a_n = s_n^2 - 1$ where s_n is a positive integer. Next, write the prime factorisation of $a_1 a_2 \cdots a_n$ as

$$p_1^{b_1(n)} p_2^{b_2(n)} \cdots p_N^{b_N(n)}$$

where $p_1, p_2, \ldots, p_N$ are the prime numbers less than 2016 and $b_1(n), b_2(n), \ldots, b_N(n)$ are non-negative integers. Define the function $f : \mathbb{N} \to \{0,1\}^N$ by

$$f(n) = \big(c_1(n), c_2(n), \ldots, c_N(n)\big)$$

where $c_i(n) = 0$ if $b_i(n)$ is even, and $c_i(n) = 1$ if $b_i(n)$ is odd. Since each c_i can be either 0 or 1 - two choices - and there are N places to choose, the number of possibilities for $f(n)$ is 2^N. Consider the last $2^N + 1$ products $a_1 a_2 \cdots a_n$, for $n = 2^{2016} - 2^N, 2^{2016} - 2^N + 1, \ldots, 2^{2016}$. By the Pigeonhole Principle, we can find at least two values $f(t)$ and $f(u)$ that are identical where $2^{2016} - 2^N \le t < u \le 2^{2016}$. Since

$$\begin{aligned} f(u) &= \big(c_1(u), c_2(u), \ldots, c_N(u)\big) \\ \text{and} \qquad f(t) &= \big(c_1(t), c_2(t), \ldots, c_N(t)\big)\,, \end{aligned}$$

are the same, we see that $c_i(t) = c_i(u)$ for all $i = 1, 2, \ldots, N$. By definition, this means that $b_i(t)$ and $b_i(u)$ are both even or are both odd; therefore, $b_i(u) - b_i(t)$ is even for all $i = 1, 2, \ldots, N$. We can therefore write $b_i(u) - b_i(t) = 2k_i$ for some integer k_i; then

$$\begin{aligned} a_{t+1} \cdots a_u &= \frac{a_1 a_2 \cdots a_u}{a_1 a_2 \cdots a_t} \\ &= \frac{p_1^{b_1(u)} p_2^{b_2(u)} \cdots p_N^{b_N(u)}}{p_1^{b_1(t)} p_2^{b_2(t)} \cdots p_N^{b_N(t)}} \\ &= p_1^{b_1(u)-b_1(t)} p_2^{b_2(u)-b_2(t)} \cdots p_N^{b_N(u)-b_N(t)} \\ &= p_1^{2k_1} p_2^{2k_2} \cdots p_N^{2k_N} \\ &= \left(p_1^{k_1} p_2^{k_2} \cdots p_N^{k_N}\right)^2 . \end{aligned}$$

Since $a_{t+1} \cdots a_u$ is an integer, each k_i is a non-negative integer, and so $a_{t+1} \ldots a_u$ is a perfect square!

How does this help? Define $a = s_u$, $b = s_t$ and $c = p_1^{k_1} p_2^{k_2} \cdots p_N^{k_N}$; then

$$\frac{a^2 - 1}{b^2 - 1} = \frac{s_u^2 - 1}{s_t^2 - 1} = \frac{a_1 a_2 \cdots a_u}{a_1 a_2 \cdots a_t} = a_{t+1} \cdots a_u = c^2 .$$

To complete the proof, let us assume that the statement to be proved is wrong: that is, let us assume that each of the numbers $a_1, a_2, \ldots, a_{2^{2016}}$ is greater than 1. Then $c > 1$, so

$$a^2 = (b^2 - 1)c^2 + 1 = b^2c^2 - c^2 + 1 < (bc)^2 .$$

Therefore, $bc > a$, so $bc \geq a + 1$. Hence,

$$\begin{aligned} c^2 - 1 &= b^2c^2 - a^2 \\ &\geq (a+1)^2 - a^2 = 2a + 1 > a > \sqrt{a^2 - 1} = \sqrt{a_1 a_2 \cdots a_u} \geq \sqrt{2^u} > 2^{2^{2015} - 2^{N-1}} . \end{aligned}$$

However, $u - t \leq 2^N$, so

$$c^2 - 1 = a_{t+1} \cdots a_u - 1 < 2016^{u-t} < 2048^{2^N} = 2^{11 \times 2^N}$$

Therefore,

$$2^{2^{2015} - 2^{N-1}} < c^2 - 1 < 2^{11 \times 2^N}$$

so

$$2^{2015} - 2^{N-1} < 11 \times 2^N .$$

It follows that

$$2^{2015} < 23 \times 2^{N-1} < 32 \times 2^{N-1} = 2^{N+4} .$$

Thus, $2015 < N+4$, so $N > 2011$. But no even number except for 2 is prime, eliminating $4, 6, \ldots, 2014$ from being prime. Since N is the number of primes at most 2016, this gives $N \leq \frac{2016}{2} + 1 = 1009$, so $2011 < N \leq 1009$, a contradiction! Thus, our assumption is incorrect, and $a_n = 1$ for some $n \in \{1, 2, \ldots, 2016\}$. $\square$

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